UNIVERSITY OF TORONTO

Faculty of Arts and Science

APRIL–MAY EXAMINATIONS 2003

PHY 252H1S

Duration – 3 hours

Aids allowed: Nonprogrammable calculators, plus one letter-format handwritten aid-sheet.

Important: All questions carry equal weight. Make sure to answer all questions.

1.) Provide a short description of the phenomenon of Bose-Einstein condensation. Your text should include the following: what kind of particles can undergo this effect and in what temperature regime it can occur; the distribution function of the particles over orbitals; what can be said about the size and sign of their chemical potential, and how this affects condensation. **Note:** Keep your answer short and to the point; the marker will *not* grade or even look beyond the first page of texts which exceed one page in a booklet. And do remember to define any symbol you use.

2.) Under certain assumptions, such as an isothermal atmosphere (known to be valid over a large range of altitudes) and a uniform gravitational acceleration g, it is possible to derive an expression for the concentration n of air molecules as a function of altitude z: $n(z) = n_0 \mathrm{e}^{-mgz/\tau}$, where m is the mass of the molecules. Yet, one can exhibit the limitations of this treatment by relaxing the assumption of uniform g.

(a) Making sure to *state explicitly any other assumption you make* beyond that of an isothermal atmosphere, obtain an expression for the concentration $n(r)$, where r is the distance to the centre of the Earth. The potential energy of an air molecule is now $-GM/r$, with G the universal gravitational constant, M the mass of the Earth. Express $n(r)$ as a function of the gravitational acceleration at the surface of the Earth, $g = GM/R^2 = 9.8\,\mathrm{m/s^2}$, where R is the radius of the Earth.

(b) Assuming spherical symmetry, write an integral giving the total number N of air molecules in the atmosphere. Without loss of generality, take the upper limit on r to be infinity. Do not attempt to evaluate the integral. Is N a well-defined number? Discuss briefly the implications of your answer with respect to assumptions you may have made.

(c) In an assignment, you have assumed instead an isentropic atmosphere where pressure p and temperature τ are related by

$$\frac{\mathrm{d}p}{p} = \frac{\gamma}{\gamma - 1}\,\frac{\mathrm{d}\tau}{\tau}, \qquad \gamma = \frac{C_\mathrm{p}}{C_\mathrm{V}} = \frac{7}{5} \text{ is the adiabatic index of air}$$

You obtained an expression for the temperature gradient valid under the assumption of uniform g. With this assumption relaxed, the expression reads:

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = \frac{1-\gamma}{\gamma}\,\frac{mgR^2}{r^2}$$

where now $g = GM/R^2$.

Using the above displayed equations, find $\tau(r)$ and then $n(r)$ under such isentropic conditions. Write the integral for N, and note that this treatment makes sense only over a *finite* range of r, so that the upper limit of integration cannot extend all the way to infinity. What is the high cut-off value of the *altitude* when ground temperature is 27°C, given that $m = 4.3 \times 10^{-26}$ kg? Again, do not compute N.

3.) Consider two samples of ideal gases which initially are at the same temperature and in equal but separate volumes, both containing N atoms. The two gases are now placed in diffusive contact, and each gas now has unimpeded access to the full combined volume, $2V$.

(a) Calculate the change in the total entropy after equilibrium is regained if the two gases are made of identical atoms.

(b) Calculate the change in the total entropy after equilibrium is regained if the two gases are made of different species of atoms.

(c) Give a *qualitative* explanation of any difference (or lack thereof) between your results in part (a) and part (b).

4.) At sufficiently low temperature, the (constant-volume) heat capacity of phonons in a crystal and of blackbody electromagnetic radiation have the same dependence on temperature.

(a) A crystal has a Debye temperature of 100 K. When it is at the same temperature, 2.7 K, as the cosmic background radiation, it contains 1.0×10^{22} atoms per cm^3. What is the ratio of its heat capacity to that of cosmic blackbody radiation occupying an equal volume?

(b) The crystal is transported to the centre of the Sun, at a temperature of 1.5×10^7 K, where it is a monatomic gas with a pressure of 1.0 MPa. Now calculate the ratio of its heat capacity at constant volume to that of an equal volume of photons, also located at the centre of the Sun. Justify any assumption you may have made.

5.) When a spin-$\frac{1}{2}$ particle is immersed in a uniform magnetic field $\boldsymbol{B}$, its total energy is equal to its kinetic energy plus a potential energy $U = -\,\boldsymbol{m} \cdot \boldsymbol{B}$, with $\boldsymbol{m}$ the magnetic moment of the particles. Thus, $U_\uparrow = -mB$ when the spin is aligned with the field (spin-up), and $U_\downarrow = mB$ when it is opposite (spin-down). In chapter 3 of Kittel and Kroemer, it is shown that a system of N such *independent* spins at temperature τ has a magnetization $M = m(N_\uparrow - N_\downarrow) = Nm\,\tanh(mB/\tau)$. As $\tau \to 0$, $M \to Nm$: all the spins are aligned with $\boldsymbol{B}$.

When the particles are conduction electrons at room temperature, however, a calculation of the magnetization must take into account the effect of the Pauli exclusion principle. Even in a 2 T-field, $U \approx \pm 10^{-4}$ eV, compared to a zero-field Fermi energy $\epsilon_\mathrm{F} \approx 5$ eV. To perform this calculation, we write for the number of spin-up electrons at $\tau = 0$:

$$N_\uparrow(\mu) \;=\; \int_{-mB}^{\mu} \mathcal{D}_\uparrow(\epsilon)\,\mathrm{d}\epsilon$$

where ϵ is the *total* energy of an electron, μ is the chemical potential of the system at $\tau = 0$, and the density of states for spin-up electrons is, to a very good approximation:

$$\mathcal{D}_\uparrow(\epsilon) = \frac{N}{2}\left[\frac{3}{2}\frac{(\epsilon + mB)^{1/2}}{\epsilon_\mathrm{F}^{3/2}}\right]$$

(a) Explain the origin of the offsets $\pm mB$ in the lower limit of integration and in the density of states. Then write a similar expression for $N_\downarrow(\mu)$ and $\mathcal{D}_\downarrow(\epsilon)$.

(b) Calculate $N_\uparrow(\mu) + N_\downarrow(\mu)$. Write $\mu = \epsilon_\mathrm{F}(1 + \delta\mu/\epsilon_\mathrm{F})$, and expand your answer to quadratic order in $(\delta\mu \pm mB)/\epsilon_\mathrm{F}$. Then implement the constraint $N = N_\uparrow(\mu) + N_\downarrow(\mu)$ in order to estimate $\delta\mu/\epsilon_\mathrm{F}$ for the typical values of mB and ϵ_F given above. Hint: in the last step, you can consistently drop $(\delta\mu/\epsilon_\mathrm{F})^2$.

(c) Derive an expression for the magnetization to *first order* in mB of the system of N conduction electrons. Your result in part (b) shows that, to a sufficiently good approximation, you can take $\mu = \epsilon_\mathrm{F}$.

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SOME POSSIBLY USEFUL EXPRESSIONS

$$\mathrm{d}U = \tau\,\mathrm{d}\sigma - p\,\mathrm{d}V + \mu\,\mathrm{d}N \qquad C_{\mathrm{V}} = \left(\frac{\partial U}{\partial \tau}\right)_{V,N} \qquad pV = N\tau$$

$$P_i V_i^{\gamma} = P_f V_f^{\gamma} \qquad \sigma = \ln g \qquad C_{\mathrm{p}} = C_{\mathrm{V}} + N$$

$$U = \frac{3}{2}N\tau \qquad P = \frac{2}{3}\frac{\langle U\rangle}{V} \qquad \sigma = N\left[\ln\frac{n_{\mathrm{Q}}}{n} + \frac{5}{2}\right]$$

$$Z = \sum_s e^{-\epsilon_s/\tau} \qquad \langle X\rangle = \sum_j X_s\, e^{-\epsilon_s/\tau} \qquad U = \tau^2\frac{\partial \ln Z}{\partial \tau} \qquad p = \tau\frac{\partial \ln Z}{\partial V}$$

$$Z = \frac{Z_1^N}{N!} \qquad Z_1 = n_{\mathrm{Q}}V \qquad n_{\mathrm{Q}} = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2} \qquad \sigma = \frac{\langle U\rangle}{\tau} + \ln Z$$

$$U = -\boldsymbol{m}\cdot\boldsymbol{B} \qquad F = U - \tau\sigma = -\tau\ln Z \qquad \mu = \left(\frac{\partial F}{\partial N}\right)_{V,\tau} = -\tau\ln(Z_1/N)$$

$$k_{\mathrm{B}} = 1.38\times10^{-23}\,\mathrm{J/K} \qquad \hbar = 1.06\times10^{-34}\,\mathrm{J\cdot s} \qquad 1\,\mathrm{eV} = 1.602\times10^{-19}\,\mathrm{J}$$

$$(U/V)_{\mathrm{Planck}} = \frac{\pi^2}{15\hbar^3c^3}\tau^4 \qquad \frac{hc}{\lambda_{\mathrm{max}}} = 4.965\,\tau \qquad \mathcal{D}_{\mathrm{em}}(\omega) = \frac{V}{\pi^2c^3}\omega^2$$

$$\langle N(\omega)\rangle_{\mathrm{Planck}} = \frac{1}{e^{\hbar\omega/\tau} - 1} \qquad \langle N\rangle = \int_0^{\infty}\langle N(\omega)\rangle\mathcal{D}_{\mathrm{em}}(\omega)\,\mathrm{d}\omega \qquad \langle X\rangle = \int_0^{\infty} X\,\langle N(\omega)\rangle\mathcal{D}_{\mathrm{em}}(\omega)\,\mathrm{d}\omega$$

$$J_{\mathrm{rad}} = \sigma_{\mathrm{B}}T^4 \qquad \sigma_{\mathrm{B}} = 5.67\times10^{-8}\,\mathrm{W/(m^2\cdot K^4)} \qquad \tau_{\mathrm{Debye}} = \hbar v\left(6\pi^2\frac{N}{V}\right)^{1/3}$$

$$f_{\mathrm{FD}}(\epsilon) = \frac{1}{\mathrm{e}^{(\epsilon-\mu)/\tau} + 1} \qquad f_{\mathrm{BE}}(\epsilon) = \frac{1}{\mathrm{e}^{(\epsilon-\mu)/\tau} - 1} \qquad C_{\mathrm{V}}(\tau \ll \tau_{\mathrm{Debye}}) \approx \frac{12\pi^4N}{5}\left(\frac{\tau}{\tau_{\mathrm{Debye}}}\right)^3$$

$$\epsilon_{\mathrm{F}} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \qquad \mathcal{D}(\epsilon) = (2s+1)\frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\epsilon^{1/2} = \frac{3N}{2}\frac{\epsilon^{1/2}}{\epsilon_{\mathrm{F}}^{3/2}}$$

$$C_{\mathrm{V}}^{\mathrm{el}} = \frac{\pi^2}{2}N\left(\frac{\tau}{\epsilon_{\mathrm{F}}}\right) \qquad \mu(\tau) = \epsilon_{\mathrm{F}}\left[1 - \frac{\pi^2}{12}\left(\frac{\tau}{\epsilon_{\mathrm{F}}}\right)^2\right]$$

$$\langle N_0\rangle_{\mathrm{BE}} = N\left[1 - \left(\frac{T}{T_{\mathrm{BE}}}\right)^{3/2}\right] \qquad \tau_{\mathrm{BE}} = \frac{2\pi\hbar^2}{m}\left(\frac{N}{2.612\,V}\right)^{2/3}$$

$$(1+x)^n \underset{x\ll1}{\approx} 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$$