University of Toronto, Faculty of Arts and Sciences

PHY252S Thermal Physics

Midterm Exam, Friday 1st March 2002

Examiner: Prof. A. Peet

Duration: 50 minutes.

Aids permitted: calculator, single-side handwritten aid sheet.

INSTRUCTIONS:

This exam has three questions and five pages.
Do ALL three questions.

Write all answers inside the single exam book.
Use the back of the test paper pages for scratch paper.
Write clearly, using either black or blue pen or 2B-or-darker pencil.
Unclear answers will *not* be marked.

Show your working to enable partial credit.

Apportion your time strictly according to marks (50 marks, 50 minutes).
Marks are indicated by square brackets next to questions in bold,
and next to parts of questions in italics.

Relevant physical constants and integrals can be found in the appendix.

Notes regarding Question 1 multiple choice:
(a) Part marks may be given for partially correct answers.
(b) Marks will be adjusted for the statistics of wild guessing (a wrong answer on a question with 5 options will incur a penalty of -1/5 of the question's marks).

GOOD LUCK!

[15] QUESTION 1: MULTIPLE CHOICE.
In each part, select the response (i)-(v) which *best* completes the statement or answers the question.

[3] 1A: The *Fundamental Assumption of Thermal Physics* is:
(i) that thermal physics is about explaining macroscopic properties starting from microphysics;
(ii) engraved on Boltzmann's tombstone;
(iii) that a closed system is equally likely to be in any quantum state accessible to it ;
(iv) that with a big enough computer, everything can be derived from string theory;
(v) that a closed system has constant energy, particle number, volume and external fields.

[3] 1B: Which of the following will tend to *increase* the entropy?:
(i) removing a constraint internal to a system;
(ii) straightening a folded protein;
(iii) breaking apart molecules;
(iv) lowering the energy;
(v) only two of the above.

[3] 1C: In deriving the *Boltzmann Factor*, the techniques used are which of the following?:
(i) the physics fact that the reservoir $\mathcal{R}$ is big by comparison to the system $\mathcal{S}$;
(ii) the math formula for a Taylor expansion;
(iii) the physics fact that in calculating the relative probability for the system $\mathcal{S}$ to be in two different quantum states, only the multiplicity function of the reservoir $\mathcal{R}$ matters;
(iv) the math fact that mixed partial derivatives are equal (e.g. $\partial^2 F/\partial\tau\partial V = \partial^2 F/\partial V\partial\tau$);
(v) only the first three of these.

[3] 1D: Thermal imaging can be used in medical physics to record heat patterns associated with tumours. If the normal human body temperature is 37°C, at about what wavelength should a photon detector have greatest sensitivity? (*Hint:* You can pretend we are blackbodies for the purposes of this quick calculation. You may find useful the number $hc/k_B \simeq 1.44\times10^{-2}$m.K.)
(i) 100nm ;
(ii) 1μm ;
(iii) 10 μm ;
(iv) 100μm ;
(v) 1mm .

[3] 1E: Two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are put in thermal contact and allowed to exchange particles. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are in equilibrium, which of the following variables must be equal for $\mathcal{S}_1$ and $\mathcal{S}_2$?:
(i) temperature;
(ii) pressure;
(iii) chemical potential;
(iv) temperature and chemical potential;
(v) temperature and Helmholtz free energy.

[15] QUESTION 2: ONE-DIMENSIONAL FERROMAGNET.

Consider a one-dimensional system of $N \geq 2$ distinguishable magnets, where each magnet can either point up (↑) or down (↓). You can picture the magnets as equally spaced along a line. Now suppose that the only important energy in the problem comes from the *nearest-neighbour pair interaction*, which has energy $\pm J$.

The situation for $N=2$ is as follows:

$$\begin{aligned} \uparrow\downarrow \text{ or } \downarrow\uparrow : \quad & \mathcal{E} = +J \quad (\text{anti}-\text{alignment})\,; \\ \uparrow\uparrow \text{ or } \downarrow\downarrow : \quad & \mathcal{E} = -J \quad (\text{alignment})\,. \end{aligned}$$

Let $J>0$, so that the magnets prefer to *align* with their neighbours. This is called ferromagnetic behaviour.

The total energy for the line of N magnets is simply the sum of nearest-neighbour pair interaction energies $\pm J$. There are $N-1$ nearest-neighbour pairs in a line of N magnets. For example, the energy of ↑↓↑↓ is $+3J$.

[4] (a) For $N=3$, three magnets, work out the possible configurations of the magnets (there are $2^N = 8$ of them). Using the above nearest-neighbour rule, find the system energy $\mathcal{E}$ for each of the eight configurations.

[5] (b) Now let us put this system in thermal contact with a reservoir.
Find the partition function for $N=3$. Show explicitly that it is

$$Z = 4\left[1 + \cosh\left(2J/\tau\right)\right]\,.$$

From Z, compute the average energy U. What are the low-τ and high-τ limits of U? Explain physically what is happening to the ferromagnet at high temperature.

[4] (c) Keep working with $N=3$ and suppose that $J = 0.025eV$. Using your friend the Boltzmann Factor, find the probability of populating the lowest energy level of the system at room temperature 300K. Repeat for the second lowest energy level of the system.

[2] (d) Does there exist a temperature at which we would get a higher population in the second energy level than the first?

[20] QUESTION 3: TWO-DIMENSIONAL PHOTON GAS.

Suppose we were flatlanders, living only in two dimensions. How would we know this, if we couldn't see into the third dimension? The answer is that we'd need to do an experiment...

Let's consider a two-dimensional box of photons, of volume $A = L^2$. Using the trial electric field $\vec{E} = (E_x, E_y)$ (with $\vec{E}_0 \cdot \vec{n} = 0$)

$$E_x = E_{x0}\ \sin(\omega_n t) \cos(n_x \pi x/L) \sin(n_y \pi y/L)\ ,$$
$$E_y = E_{y0}\ \sin(\omega_n t) \sin(n_x \pi x/L) \cos(n_y \pi y/L)\ ;$$

and the Maxwell equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\vec{E} = 0\,,$$

it can be shown that the allowed frequencies of the system are given by

$$\frac{\omega_{\vec{n}}^2}{c^2} = \left(\frac{\pi}{L}\right)^2 \left(n_x^2 + n_y^2\right)$$

[4] (a) What are the two lowest (nonzero) energies allowed in this system? What is the difference between them, $\Delta\mathcal{E}$?

In the next part of this question, we are going to replace a sum over $\vec{n}$ by an integral over n_x, n_y. To justify this, we'll need for $x \equiv \mathcal{E}/\tau$ to be essentially a continuous variable. For the sake of argument, let us demand that $\Delta\mathcal{E}/\tau < 1 \times 10^{-6}$. Take a box of side L=10m and figure out the temperatures T for which this condition is satisfied.

[8] (b) Starting from the partition function for a single mode of frequency ω_n in the box

$$Z_{\vec{n}} = \frac{1}{1 - \exp(-\mathcal{E}_{\vec{n}}/\tau)}$$

find the free energy $F_{\vec{n}}$ for that particular mode. Integrate $F_{\vec{n}}$ over all modes labelled by $\vec{n}$ to find the total free energy F. You'll need to use the fact that in two dimensions there is only *one* independent photon polarisation. You'll also need the integrals in the appendix, and it will help to change variables to $x = (\hbar n c\pi)/(L\tau)$. You should get the answer

$$F = -\frac{\zeta(3)}{2\pi}\frac{A\tau^3}{(\hbar c)^2}\,.$$

[5] (c) Using F, compute the entropy σ. Also, using the thermodynamic relation $F = U - \tau\sigma$, find the average energy U.

[3] (d) Use your results on U to find the specific heat at constant volume C_A. Draw a rough graph of C_A as a function of temperature, and interpret. What is the essential difference from three dimensions?

APPENDIX: Some potentially useful constants and formulæ

$h = 6.63 \times 10^{-34}$J.s ; $c = 3.00 \times 10^{8}$m.s^{-1} ; $k_B = 1.38 \times 10^{-23}$J.K^{-1} ; $e = 1.60 \times 10^{-19}$C.

$\hbar = h/(2\pi)$

Rectangular to plane polar coordinates:

$$\int_{-\infty}^{+\infty} dn_x \int_{-\infty}^{+\infty} dn_y = \int_0^{\infty} dn \int_0^{2\pi} n d\phi$$

Integration:

$$\int_0^{\infty} dx\ x\ \log(1 - e^{-x}) = -\zeta(3)$$

and

$$\int_0^{\infty} dx \frac{x^n}{e^x - 1} = n!\ \zeta(n{+}1)$$

Photons:

$$\mathcal{E} = \hbar\omega = \frac{hc}{\lambda}$$