University of Toronto, Faculty of Arts and Science
PHY252S Thermal Physics
Final Exam, Monday 22nd April 2002
Examiner: Prof. A. Peet
Duration: 3 hours.

Aids permitted: calculator, double-side handwritten aid sheet.

Do *both* questions in Part A, and *three of four* questions in Part B.
Each question is worth 20 points.

You have three hours and five problems to do.
Apportion your time sensibly: spend about half an hour per problem.

Write all answers inside the exam book.
Write clearly, using either black or blue pen or 2B-or-darker pencil.
Unclear answers will *not* be marked.

Show your working to enable partial credit.
(e.g. catch a math error using physical reasoning.)

Useful formulæ and physical constants can be found on page 7.

PART A: Do *both* questions.

Q1: Qualitative

Explain the following terms briefly, illustrating your understanding of thermal physics. Write no more than three sentences for each:

[2] (1a) Fundamental Assumption of Thermal Physics

[2] (1b) Reservoir

[2] (1c) Canonical ensemble

[2] (1d) Gibbs sum

[2] (1e) Orbital

Explain the following phenomena with an example, showing that you understand the connection between microscopics and macroscopic thermodynamics. Write no more than 2 paragraphs for each:

[5] (1f) Bose-Einstein condensation.

[5] (1g) The Equipartition Theorem, i.e. the fact that at high enough temperatures, the energy U is $\frac{1}{2}N\tau$ per degree of freedom (also explain what happens at lower temperature).

Q2: Multiple choice

Note: choose the option that *best* answers each question.
Partial credit for partially correct answers; *No* penalty for wrong answers.

[2.5] (2a) A Maxwell relation is:
(i) a statement relating heat, work, and internal energy.
(ii) an equation of motion for the electromagnetic field.
(iii) a law of blackbody radiation.
(iv) a theorem about Maxwell demons.
(v) a physical relation derived from mathematical equality of mixed second partial derivatives.

[2.5] (2b) Suppose that the universe consisted of only photons after the Big Bang. Make the approximation that the expansion of the universe since then was isentropic and reversible. If the universe now is $\sim 10^3$ times bigger than at some previous time, how hot was the universe then compared to now?
(i) About 10^{12} times hotter.
(ii) About 10^9 times hotter.
(iii) About 10^6 times hotter.
(iv) About 10^3 times hotter.
(v) None of the above, it was colder back then.

[2.5] (2c) Consider a reversible isentropic expansion of an ideal gas. If the volume is decreased,
(i) p increases, but U, τ stay the same.
(ii) only p, τ increase; U stays the same.
(iii) p, U increase, but τ stays the same.
(iv) p, τ, U all increase.
(v) None of the above is true.

[2.5] (2d) Quantum concentration in thermal physics is:
(i) the intermolecular spacing in a Bose-Einstein condensate, in units of the thermal de Broglie wavelength of the boson.
(ii) the concentration of defects in a crystal where we apply the Debye theory of phonons.
(iii) a parameter used to describe both ideal gases and electrons in a metal.
(iv) the concentration of gas at which the thermal de Broglie wavelength is equal to the size of the box the gas is in.
(v) exactly two of the above are correct.

[2.5] (2e) Which are examples of *external* chemical potential?
1: the potential of electrolyte ions in the electric field throughout the inside of a battery;
2: the potential of fixed electric dipoles in a non-uniform electric field;
3: the potential of monatomic gas atoms in a gravitational field;
4: the potential of pressure differential in a fluid across a semi-permeable membrane (osmotic pressure);
5: the potential of carbon monoxide binding chemically to hæmoglobin.
(i) 1,2,3
(ii) 2,3,5
(iii) 2,3,4
(iv) 1,3,5
(v) none of 1-5.

[2.5] (2f) Treat Mars and Earth as approximately blackbodies. If Mars orbits at a radius from the Sun of about 1.5 times that of Earth, and Earth's temperature is about 300K, how cold is it on Mars?
(i) about 133K.
(ii) about 200K.
(iii) about 245K.
(iv) about 271K.
(v) about 300K.

[2.5] (2g) In the Debye theory of solids, the heat capacity as a function of absolute temperature τ is:
(i) linear at low τ , and cubic at high τ.
(ii) linear at low τ, and constant at high τ.
(iii) quadratic at low τ, and constant at high τ.
(iv) cubic at low τ, and constant at high τ.
(v) quartic at low τ, and linear at high τ.

[2.5] (2h) The Fermi energy ε_F is:
(i) The energy equivalent to the rest mass of the fermion.
(ii) The chemical potential of a Fermi gas at absolute zero.
(iii) The thermal energy at which fermions become able to crowd into the same quantum state.
(iv) k_B times the temperature at which electrons in white dwarf stars become relativistic.
(v) The energy around which conduction electrons in metals give rise to a sizeable quadratic contribution to the metal's heat capacity.

PART B: Do *three of four* questions.

Q3: The Spins and the Wire

Consider a gas made up of N little spins, which can either be aligned to a magnetic field ($\uparrow$) or anti-aligned ($\downarrow$). Now introduce a current-carrying wire, which creates a magnetic field in the azimuthal direction $\vec{e_\phi}$:

$$\vec{B}(\rho) = \frac{I}{2\pi\rho}\vec{e_\phi}$$

Here ρ is the distance from the wire, which has radius ρ_0, and I is a constant.

Now recall that $U = -\vec{m}\cdot\vec{B}$ is the alignment energy. It follows that the $\uparrow$-type and $\downarrow$-type particles can be thought of as different subspecies of gas; they have different external chemical potentials $\mu_{\text{ext}} = \mp mB(\rho)$.

Assume that the gas is ideal, even when the wire is present.

[5] (3a) What is the *internal* chemical potential of the gas particles? (You may assume that the presence of the wire does not muck up the partition function of the gas particles, and that the gas lives in the usual cubic box of side length L).

[5] (3b) Draw a little picture of what the two species look like in the $\vec{B}$ field. What are the two conditions for thermal equilibrium between the $\uparrow$-type and $\downarrow$-type particles?

[5] (3c) Find the concentrations for the two species: $n_\uparrow(\rho)$ and $n_\downarrow(\rho)$.

[5] (3d) Find the total concentration $n(\rho) = n_\uparrow(\rho) + n_\downarrow(\rho)$. Intepret your results of (c,d) in words and/or graphically.

Q4: Radiation

Imagine you have two infinite square plates at temperatures T_1, T_2. Assume they are in vacuum. Take all plates in this problem to be parallel.

[5] (4a) Assuming the plates are blackbodies, find the net flux of photons between the two plates.

[5] (4b) Next, put a third identical plate between the other two plates. Work out the temperature T_3 of this third plate such that it is in equilibrium with the other two. Show that the net flux between plates 1 and 2 is reduced by a factor of two by putting this third plate in between.

[5] (4c) This time, put two separate extra plates in between. Do a similar computation to work out how much the flux is reduced with this configuration. Then guess what happens in general.

[5] (4d) Work out what happens if there is only one plate in between, and it has absorptivity and emissivity e instead of 1.

Congratulations, you have just designed a heat shielding system for transporting low-temperature stuff like liquid nitrogen.

Q5: Number fluctuations

[8] (5a) Starting from the grand partition function $\mathcal{Z}$ (curly Z), and the basic definition of the thermal average of a quantity, show that

$$\langle N \rangle = \frac{\tau}{\mathcal{Z}} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)_{\tau,V} \quad \text{and} \quad \langle N^2 \rangle = \frac{\tau^2}{\mathcal{Z}} \left(\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} \right)_{\tau,V}$$

[2] (5b) Use the above to show that the mean square fluctuation in number is

$$\langle (\Delta N)^2 \rangle = \tau^2 \left[\frac{1}{Z} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \frac{1}{Z^2} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)^2 \right]$$

[8] (5c) By studying a single orbital, use the Fermi-Dirac and Bose-Einstein distribution functions to show that the fractional fluctuations are

$$\frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle^2} = \left(\frac{1}{\langle N \rangle} \mp 1 \right)$$

Interpret. Be sure to indicate which sign belongs to fermions and which to bosons.

[2] (5d) What happens to the above result when the gas becomes classical? Why?

Q6: Two-dimensional Fermi Gas

Consider a monatomic gas of N spin-s fermions, in a box of area $A = L^2$. Work at absolute zero temperature, i.e. in the extreme quantum regime.

[6] (6a) Treat the gas non-relativistically, i.e. take the energy levels appropriate to $\vec{n}$ to be

$$\varepsilon_{\vec{n}}^{\mathrm{NR}} = \frac{|\vec{p}|^2}{2M}$$

where the momentum is

$$|\vec{p}| = \frac{\hbar\pi|\vec{n}|}{L}$$

Show that the nonrelativistic Fermi energy is

$$\varepsilon_F^{\mathrm{NR}} = \frac{1}{(s+\frac{1}{2})}\frac{\hbar^2\pi}{M}\frac{N}{A}$$

(Hints: Use the fact that in two dimensions for large numbers of particles

$$\sum_{\vec{n}} \longrightarrow \frac{1}{4}\cdot(2s+1)\cdot\int dn\int n d\phi$$

where $n = |\vec{n}|$. Start by filling up to n_F such that the total number of particles is N. Also, don't confuse yourself by using the same symbol for the magnitude of the $\vec{n}$ and the concentration! Stick with N/A for the concentration.
Significant partial credit will be awarded for getting this by dimensional analysis.)

[6] (6b) Find the energy of this degenerate non-relativistic Fermi gas. Show that it is

$$U_0^{\mathrm{NR}} = \frac{1}{2}\varepsilon_F$$

per particle.

[4] (6c) Now take the Fermi gas particles to be almost massless, so that they have to be treated relativistically. i.e. take

$$\varepsilon_{\vec{n}}^{\mathrm{R}} \simeq |\vec{p}|c$$

Using the same methods as in (a), work out the relativistic Fermi energy $\varepsilon_F^{\mathrm{R}}$.

(Hint: Significant partial credit will be awarded for getting this by dimensional analysis.)

[4] (6d) Work out the energy U_0^{R} of the relativistic degenerate Fermi gas.

Helpful formulæ

Entropy:

$$\sigma = \log g(U,N)\,,\ \frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_{V,N}\,,\ \frac{p}{\tau} = \left(\frac{\partial\sigma}{\partial V}\right)_{U,N}\,,\ \frac{-\mu}{\tau} = \left(\frac{\partial\sigma}{\partial N}\right)_{U,V}\,,\ C_V = \tau\left(\frac{\partial\sigma}{\partial\tau}\right)_V$$

Helmholtz Free Energy:

$$F = -\tau\log(Z)\,,\ \mu = \left(\frac{\partial F}{\partial N}\right)_{\tau,V}\,,\ \sigma = \left(\frac{\partial F}{\partial\tau}\right)_{V,N}\,,\ p = -\left(\frac{\partial F}{\partial V}\right)_{\tau,N}\,,$$

First law:

$$dU = \tau d\sigma - pdV + \mu dN$$

Ideal gas:

$$Z = \frac{1}{N!}(n_Q V)^N\,,\ n_Q = \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}\,,\ \sigma = N\left[\log\left(\frac{n}{n_Q}\right) + \frac{5}{2}\right]\,,\ \mu = \tau\log\left(\frac{n}{n_Q}\right)$$

and

$$pV = N\tau\,,\ U = \frac{3}{2}N\tau\,,\ \text{isentropic}: pV^\gamma = \text{const. and } \tau V^{\gamma-1} = \text{const.}\,,$$

Blackbody:

$$U = \frac{\pi^2}{15\hbar^3c^3}V\tau^4\,,\ \sigma = \frac{4\pi^2}{45\hbar^3c^3}V\tau^3\,,\ J_U = \frac{cU}{4V} = \sigma_B T^4\,,$$

Grand partition function:

$$\mathcal{Z} = \sum_{N=0}^{\infty}\sum_{s(N)}\exp\left[-\frac{\left(\varepsilon_{s(N)} - \mu N\right)}{\tau}\right]$$

Fermi-Dirac and Bose-Einstein distribution functions:

$$f(\varepsilon) = \frac{1}{\exp[(\varepsilon-\mu)/\tau] \pm 1}$$

Physical Constants:
$k_B = 1.3807 \times 10^{-23}$ JK^{-1}
$\sigma_B = 5.670 \times 10^{-8}$ Wm^{-2}K^{-4}
$c = 2.9979 \times 10^{8}$ ms^{-1}
$e = 1.6022 \times 10^{-19}$ C
$R = 8.3145$ J mol^{-1}K^{-1}
1eV $= 1.6022 \times 10^{-19}$ J
$h = 6.6261 \times 10^{-34}$ Js^{-1}, $\hbar = h/(2\pi)$