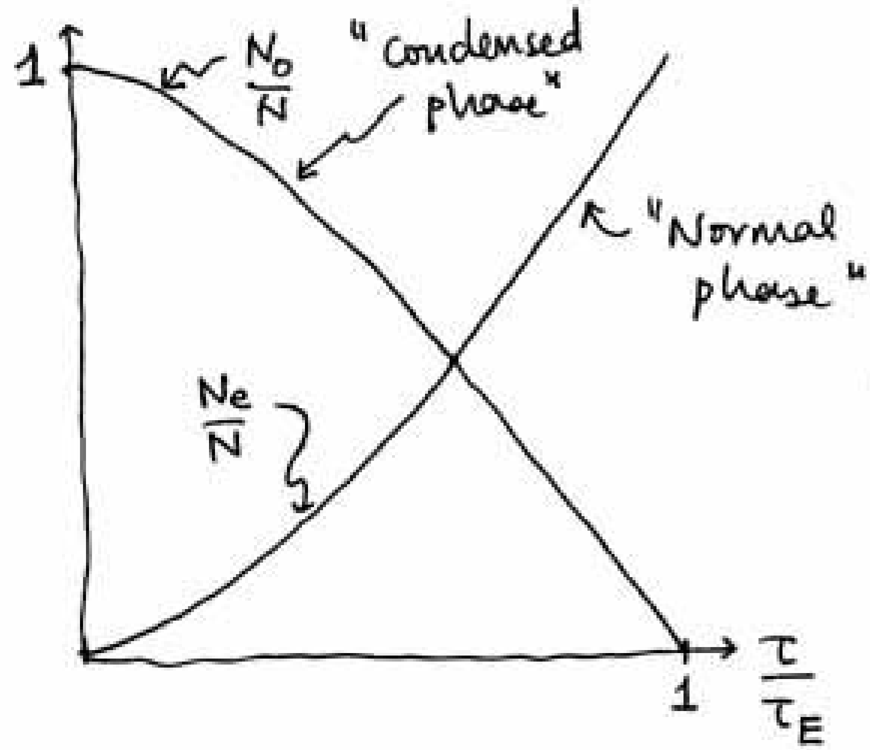
Again using the $N_0 \gg 1$ approx,
find

$$N_0 \equiv N - N_e$$
$$\cong N\left[1 - \left(\frac{\tau}{\tau_E}\right)^{3/2}\right]$$



Experimentalists (& theorists) will refer to condensed & normal phase as if they're two different substances (they do behave very differently!)

$\boxed{\text{One Example: } {}^4He}$

How come ^4He is a boson?

Nucleus has 2 protons ($\Leftrightarrow$ is Helium)
2 neutrons (4 − 2)

both protons & neutrons are (composite) spin-$\frac{1}{2}$ fermions.

Rules of angular momentum addition
say

$$s=\tfrac{1}{2} \text{ "+" } s=\tfrac{1}{2} \rightarrow \text{either } s=0 \text{ or } s=1 \quad \text{boronic}$$

or two fermions give a boson

So with $2p^+$, $2n^0$ nucleons, ^4He is a boson.

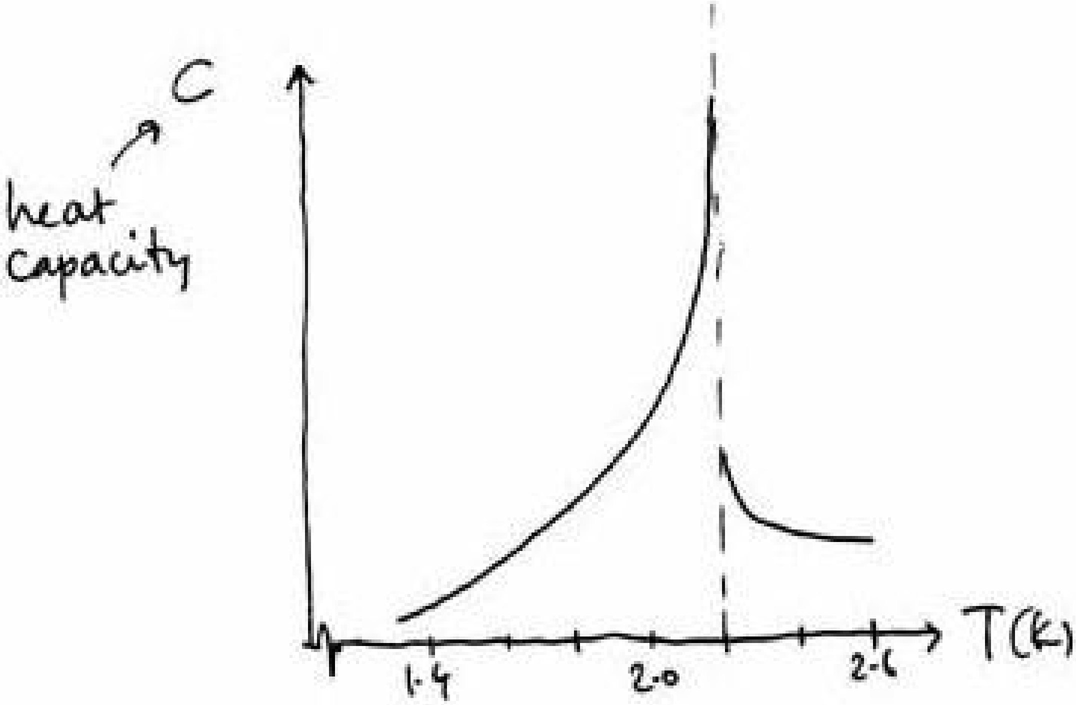
What about ^3He? This has
$2p^+$, $1n^0$ $\Rightarrow$ fermionic nucleus.

The ^3He's do a fun trick in order to get into the ground orbital – they pair up!!
Get ($^3He - {}^3He$) condensing into ground state

Actually, electrons do this trick too, at low enough τ, becoming a (e-e) "Cooper Pair" in a metal – which is a mechanism for going superconducting.

(*) In truth, the real physics of superconductors and superfluids is MUCH more complicated than our simple BEC story here ... the only thing our theory really gets right is the condensation.

For liquid ^4He,



The shape of this graph looks like a λ, so it's called the "lambda point" transition

Below T_λ, the "condensed" phase is a superfluid, which has extremely low viscosity.

A bunch of superfluid can even "climb" the walls of a (very cold) beaker! (– by capillarity...)

It is also an extremely good thermal conductor

Superconductivity & related phenomena are a HUGE area of research, both theoretical and experimental. [redacted]

About superfluid ³He

Recall that

$$T_E = \left(\frac{2\pi\hbar^2}{M}\right)\left(\frac{N}{\zeta(\frac{3}{2})V}\right)^{2/3}$$

Now, (^3He-^3He) is essentially a diatomic molecule. It is naturally going to take up more space than a single ^4He boson. Quite a bit more...

$\Rightarrow T_E(^3He\text{-}^3He) < T_E(^4He)$

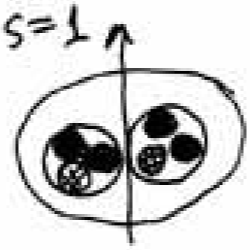
(Mass factor ↑ also helps this).

Experimentally, transition temperature ~ few mK.

* Interestingly, for the 2 spin-$\frac{1}{2}$ ³He fermions, when they pair up like this, they are in the $S=1$ angular momentum state!

$$\left(s=\tfrac{1}{2} \text{ "+" } s=\tfrac{1}{2} \rightarrow s=0 \text{ or } \underline{\underline{s=1}}\right)$$

$s=1$



Therefore, $m_s = -1, 0, +1$

and this system has magnetic properties

→ <u>FUN</u> ☺

[DOUG OSHEROFF STORY]

Back to low-temperature physics!

Brief review to refresh memory -

Fermions

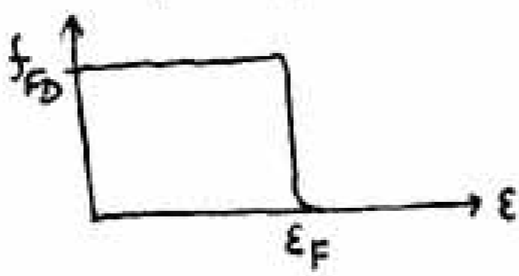
$$f_{FD} = \frac{1}{e^{(\varepsilon-\mu)/\tau} + 1}$$

$$\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

(for spin-$\frac{1}{2}$)

$$\tau_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}$$

At very low temperature $\mu \cong \varepsilon_F$ and



"Fermion elbows"

$\Leftrightarrow$ large kinetic energy even at absolute zero

Degenerate Fermi Gas

Bosons

$$f_{BE} = \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$$

$$\mathcal{D}(\varepsilon) = \frac{V}{4\pi^2}(2s+1)\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

(for spin-s)

$$\tau_E = \frac{2\pi\hbar^2}{M}\left(\frac{1}{\zeta(\frac{3}{2})} n\right)^{2/3}$$

At very low temperature

$$N(\varepsilon=0) \equiv N_0 \gg 1$$

$$N_0 \cong N\left[1 - \left(\frac{\tau}{\tau_E}\right)^{3/2}\right]$$

$$N_e \cong N\left(\frac{\tau}{\tau_E}\right)^{3/2}$$

Almost all gas particles in ground orbital ($\varepsilon = 0$)

Bose-Einstein Condensation

Number fluctuations

In HW #3, you took

$$\langle N \rangle = \frac{\tau}{\mathcal{Z}}\left(\frac{\partial \mathcal{Z}}{\partial \mu}\right)_{\tau V}$$

and derived

$$\langle N^2 \rangle = \frac{\tau^2}{\mathcal{Z}}\left(\frac{\partial^2 \mathcal{Z}}{\partial \mu^2}\right)_{\tau V}$$

The mean square deviation is

$$\langle (\Delta N)^2 \rangle \equiv \langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$$

$$= \frac{\tau^2}{\mathcal{Z}}\frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \frac{\tau^2}{\mathcal{Z}^2}\left(\frac{\partial \mathcal{Z}}{\partial \mu}\right)^2$$

which could be rewritten as

$$\boxed{\langle (\Delta N)^2 \rangle = \tau \frac{\partial}{\partial \mu}\langle N \rangle}$$

Let's apply this to fermion & boson gases! We'll keep it simple and look just at one orbital.

(Note - this is actually KK 7.11 and 7.12 ...)

So we have

Fermions

$\langle(\Delta N)^2\rangle = \langle N\rangle(1-\langle N\rangle)$

$$\frac{\langle(\Delta N)^2\rangle}{\langle N\rangle^2} = \frac{1}{\langle N\rangle} - 1$$

$$= \frac{(1-\langle N\rangle)}{\langle N\rangle}$$

≥ 0

because $0 \leq \langle N\rangle \leq 1$

For orbitals far below ε_F at low τ, fractional fluctuations are very small ($\to 0$) because $\langle N\rangle \to 1$!

Bosons

$\langle(\Delta N)^2\rangle = \langle N\rangle(1+\langle N\rangle)$

$$\Rightarrow \frac{\langle(\Delta N)^2\rangle}{\langle N\rangle^2} = \frac{1}{\langle N\rangle} + 1$$

≥ 1

because $0 \leq \langle N\rangle < \infty$

At very low τ, BEC happens, and for ground orbital $\langle N\rangle \to$ large!

But for $\langle N\rangle \to$ large, $\frac{\langle(\Delta N)^2\rangle}{\langle N\rangle^2} \sim 1$ so huge fluctuations (absolute) !

This is one more manifestation of the huge effect of spin & statistics of particles
- at really low τ it matters a lot whether you have elbows or not.

What about the classical limit?

We can do this by taking

$$f_c = \frac{1}{e^{(\varepsilon-\mu)/\tau} \cancel{\pm 1}} \rightarrow e^{-(\varepsilon-\mu)/\tau}$$

Then $\tau \frac{\partial f_c}{\partial \mu} = e^{-(\varepsilon-\mu)/\tau} = \langle N \rangle_c$

so $\frac{\langle (\Delta N)^2 \rangle_c}{\langle N \rangle_c^2} = \frac{1}{\langle N \rangle_c}$ This is the ideal gas result.

Superfluidity

Let's pretend that the condensed phase (e.g. ^4He) is describable as a Bose-Einstein condensate.

The N_0 atoms have $\varepsilon = 0$
and hence $\vec{p} = \vec{0}$

and it's almost like there's nothing there...

In terms of viscosity, these N_0 atoms don't contribute ~~att~~ all

-unless you kick one (or some) up into higher (excited) orbitals

So we're saying that friction, which you think of as a macroscopic thing, has microscopic origins.
In this case, it would correspond to (e.g.)
^{4}He ($\varepsilon = 0$) hits wall, gets deflected, changes momentum, gets excited ...

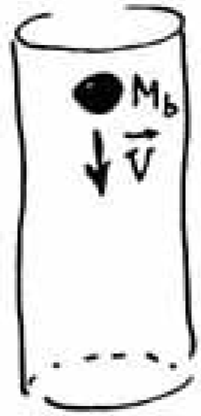
In fact, the superfluid is not a bunch of free particles. As with many other many-body systems, the superfluid has a bunch of collective modes - excitations which belong to the whole system.
In crystals these excitations are sound waves
In superfluids they are longitudinal sound waves.

These waves, quantized, are called <u>quasiparticles</u>

Quasi particles are not like electrons - they are not present if you break up the many-body system. But for low-energy physics of many-body systems working with quasiparticles is extremely productive as a framework for thinking

So how can you see this stuff in superfluids?

Get ^4He extremely cold & gone superfluid



Get a very smooth cylinder & fill it with superfluid.

Drop a smooth ball.

In order to make an excitation (quasiparticle) need to conserve energy & momentum

(1) $$\frac{1}{2} M_b \vec{v}^2 = \frac{1}{2} M_b (\vec{v}')^2 + \underbrace{\varepsilon_{\vec{k}}}_{\text{quasiparticle energy}}$$

(2) $$M_b \vec{v} = M_b \vec{v}' + \underbrace{\hbar \vec{k}}_{\text{quasiparticle momentum}}$$

Are these equations always satisfiable?

Let's get rid of $\vec{v}'$:

(2) $$\vec{v}' = \vec{v} - \frac{\hbar \vec{k}}{M_b}$$ Substitute into (1) -

$$\Rightarrow (1) \quad \frac{1}{2} M_b \vec{v}^2 = \frac{1}{2} M_b \left(\vec{v} - \frac{\hbar \vec{k}}{M_b} \right)^2 + \varepsilon_{\vec{k}}$$

i.e.

$$\cancel{\tfrac{1}{2}M_b\vec{v}^2} = \cancel{\tfrac{1}{2}M_b\vec{v}^2} + \frac{\hbar^2\vec{k}^2}{2M_b} - \hbar\vec{v}\cdot\vec{k} + \varepsilon_{\vec{k}}$$

so $$\hbar\vec{v}\cdot\vec{k} = \varepsilon_{\vec{k}} + \frac{\hbar^2\vec{k}^2}{2M_b}$$

Simplification: take M_b to be large compared to $\hbar^2\vec{k}^2$. then

$$\hbar\vec{v}\cdot\vec{k} \cong \varepsilon_{\vec{k}}$$

The meaning of this?

- if $\vec{v} = \vec{0}$, nope! No "friction"

when $\vec{v}$ reaches right amount, will satisfy eqn.

need $$\boxed{\vec{v} \parallel \vec{k} \quad \text{and} \quad v_c = \frac{\varepsilon_{\vec{k}}}{|\hbar\vec{k}|}}$$

So what happened?

faster ball $\longrightarrow$ slower ball + quasiparticle

Can think of this as a scattering process (!)

We had for phonons

$$\varepsilon_{\vec{k}} = \hbar \vec{k}\, v_s \qquad \left(\varepsilon_{\vec{k}} = \hbar\omega_{\vec{k}} ;\ \omega_{\vec{k}} = v_s \vec{k} \right.$$

speed of sound

rather than $c\vec{k}$ for photons)

So if this were the dispersion relation for quasiparticles in superfluid ^4He, we'd have

$$v_c = \frac{\varepsilon_{\vec{k}}}{|\hbar\vec{k}|} = v_s .$$

This is not actually what happens in ^4He; it's close conceptually though.

People really measure this stuff experimentally!

Last note: Unless $\tau = 0$, below τ_E there is a normal fluid component too, and this does have viscosity.

So one has to take into account bother effects.

[$v_c \approx 50\,\mathrm{ms}^{-1}$ from certain experiments ... check our large-M_0 approx.]