BOSE- EINSTEIN Condensation

$$f_{BE}(\varepsilon, \tau) = \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$$

Occupancy of ground orbital is

$$f_{BE}(0, \tau) = \frac{1}{e^{-\mu/\tau} - 1}$$

BEC is all about the fact that, at $\tau = 0$, all the bosons go into the lowest orbital!

$$N = f_{BE}(0,0)$$

So we need

$$\lim_{\tau \to 0} \frac{1}{e^{-\mu/\tau} - 1} \qquad \text{As } \tau \to 0, \quad e^{-\mu/\tau} \to \;??$$

we know $(e^{-\mu/\tau} - 1)$ must be small so that N is big. So we can use the series expansion of $e^{-\mu/\tau} \approx 1 - \frac{\mu}{\tau} + O(\text{smaller terms})$

$$\Rightarrow \quad N = -\frac{\tau}{\mu} \quad \text{or} \quad \boxed{\mu \underset{\tau \to 0}{=} -\frac{\tau}{N}}$$

In other words,

as $\tau \to 0$ μ gets very small

and is negative.

μ has to be negative as $\tau \to 0$ because

ϵ is (taken to be) zero there...

and we wouldn't want $P(\epsilon, \mu) = \dfrac{e^{-(\epsilon-\mu N)/\tau}}{\mathcal{Z}}$

to be negative!

Look at the calculation in the text

to see how amazingly much the

ground orbital is occupied as $\tau \to 0$

compared to even the first excited orbital!

The reason is essentially that μ is so

tiny : $\mu = -\frac{\tau}{N}$ is parametically small

c.f. energy spacings because it goes like $\frac{1}{N}$

(for is the reason)

Bose-Einstein Condensation

Last time, we found

$$\lim_{\tau\to 0} f_{BE}(\varepsilon=0,\tau) = N \simeq -\frac{\tau}{\mu}$$

so $\mu \xrightarrow[\tau\to 0]{} -\frac{\tau}{N}$ is parametrically small (order $\frac{1}{N}$ c.f. $\tau \neq 0$).

Density of States?

We derived $\mathcal{D}(\varepsilon)$ starting with Schrödinger eqn for non-relativistic particle; it was not specific to fermions actually. So let's "change the spin" $\cancel{s=\frac{1}{2}} \to s=0$, so there's only one spin-z component i.e. spin multiplicity $=1$ (not 2)

$$\Rightarrow \quad \mathcal{D}(\varepsilon) = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

Now we multiply this by $\underline{\underline{f_{BE}}}$ to get the available occupied orbital story-

$N = ?$

Conventional to divide up gas into

States in ground orbital $N_0(\tau)$

excited states $N_e(\tau)$

Notice that in replacing sums by integrals we've gotten ourselves a $\mathcal{D}(\varepsilon)$ which goes to zero at $\varepsilon=0$: $\mathcal{D}(\varepsilon) \propto \varepsilon^{\frac{1}{2}}$!

So we write

$$N = N_0(\tau) + \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon)\, f_{BE}(\varepsilon,\tau)$$

"Condensed Phase" "Normal Phase"

We already know $N_0(\tau)$:

$$N_0(\tau) = \frac{1}{e^{-\mu/\tau} - 1}$$

Here, μ depends on τ ;
only near $\tau \to 0$ is it $\mu \xrightarrow[\tau\to 0]{} \frac{-\tau}{N}$.

We can calculate $N_e(\tau)$:

$$N_e = \int_0^\infty d\varepsilon \left\{ \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \right\} \frac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$$

Change to $x \equiv \frac{\varepsilon}{\tau}$ $x =$ energy in units of thermal energy $\tau = k_B T$

$$\therefore N_e = \left[\tau^{3/2} \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2}\right] \int_0^\infty dx \frac{\sqrt{x}}{\left[e^{-\mu/\tau} \cdot e^{+x} - 1\right]}$$

<u>Approximation</u> : $N_0 \gg 1$

Then by $N_0(\tau) = \dfrac{1}{e^{-\mu/\tau} - 1}$

we have $e^{-\mu/\tau} \approx 1^+$

and so

$$N_e \doteq \tau^{3/2} \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty dx \, \frac{\sqrt{x}}{(e^x - 1)}}_{\zeta(\frac{3}{2})\frac{\sqrt{\pi}}{2}}$$

i.e.

$$N_e = \tau^{3/2} \frac{V}{4\pi^2} \zeta(\tfrac{3}{2})\, 2^{3/2-1} \sqrt{\pi} \left(\frac{M}{\hbar^2}\right)^{3/2}$$

roughly 2.612

$$N_e = V \zeta(\tfrac{3}{2}) \left(\frac{M\tau}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow \boxed{\frac{N_e}{N} \underset{N_0 \gg 1}{\cong} \zeta(\tfrac{3}{2}) \frac{n_Q(\tau)}{n}}$$

where $n_Q(\tau) = \left(\dfrac{M\tau}{2\pi\hbar^2}\right)^{3/2}$

$n = \dfrac{N}{V}$

Define

$$\boxed{\tau_E \doteq \frac{2\pi\hbar^2}{M} \left(\frac{N}{\zeta(\frac{3}{2}) V}\right)^{2/3}}$$

τ_E is <u>Einstein Condensation Temperature</u>

then $$\boxed{\frac{N_e}{N} \underset{N_0 \gg 1}{\cong} \left(\frac{\tau}{\tau_E}\right)^{3/2}}$$

* What does this last formula really say?

"If $N_0 \gg 1$, then $\frac{N_e}{N} \simeq \left(\frac{\tau}{\tau_E}\right)^{3/2}$"

So at $\tau \ll \tau_E$, $\frac{N_e}{N} \ll 1$

At $\tau = \tau_E$, $\frac{N_e}{N} \simeq 1$

At $\tau \gg \tau_E$, $\frac{N_e}{N} \gg 1$ — eh?? $N = N_0 + N_e$!!

Well, the problem is that we've driven our little formula outside the régime of its validity. At $\tau \sim \tau_E$, $N_e \sim N$; what this means is that essentially all the particles have left the ground orbital and gotten excited.

Above τ_E, there can still be a few particles (bosons) in the ground orbital, but <u>not</u> a macroscopic #.

* τ_E should be thought of as a <u>characteristic temperature</u>, like τ_{Fermi} or Θ_{Debye}.

* Dimensional analysis?

$\underline{\underline{T_E}}$. $T_E = \frac{2\pi\hbar^2}{M}\left(\frac{N}{\zeta(\frac{3}{2})V}\right)^{2/3}$

$$\left[\left(\frac{N}{V}\right)^{2/3}\right] = \left[V^{-2/3}\right] = [\text{length}]^{-2}$$

$[T_E] = [\text{energy}]$?

$$[\text{RHS}] = \frac{[\hbar^2]}{[M]}[\text{length}]^{-2} = \frac{[\text{energy}\cdot\text{time}]^2\,[\text{length}]^{-2}}{[\text{mass}]}$$

$$= [\text{energy}]\cdot\frac{[\text{time}]^2}{[\text{mass}]\,[\text{length}]^2}$$

But "$E = mc^2$" so dimensions work out

☺

$$T_E = \left(\frac{2\pi\hbar^2}{M}\right)\left(\frac{N}{\zeta(\frac{3}{2})V}\right)^{2/3}$$

$\propto \hbar^2 \Rightarrow$ quantum phenomenon;

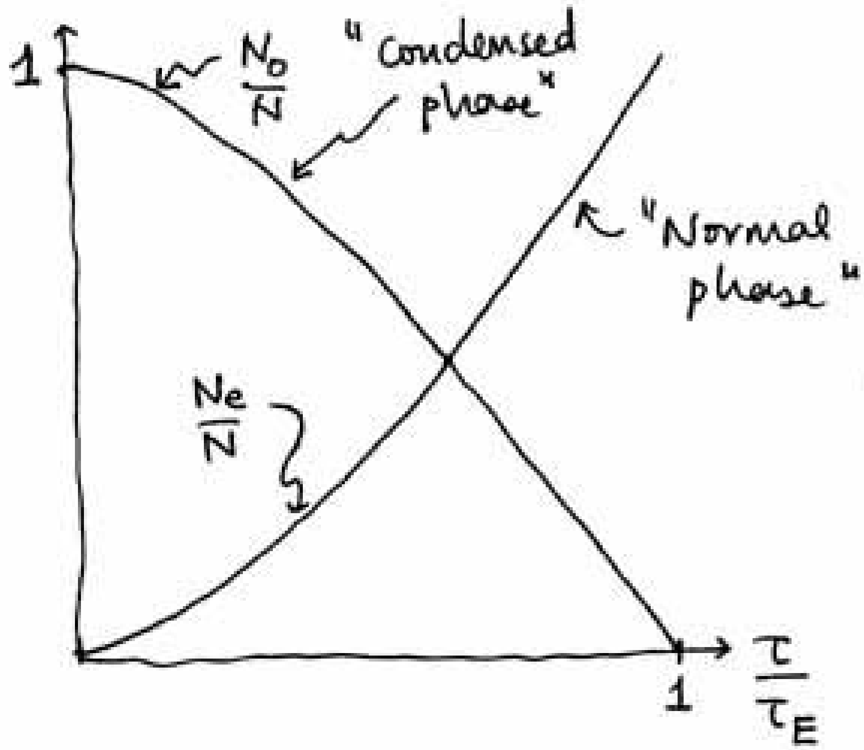
($\hbar^2/M \Leftrightarrow$ nonrelativistic boson)

$\propto \left(\frac{N}{V}\right)^{2/3} \Rightarrow$ higher T_E for denser gas of bosons

$\propto \frac{1}{M} \Rightarrow$ higher T_E for lighter bosons

Again using the $N_0 \gg 1$ approx,
find

$$N_0 \equiv N - N_e$$
$$\cong N\left[1 - \left(\frac{\tau}{\tau_E}\right)^{3/2}\right]$$



Experimentalists (& theorists) will refer to condensed & normal phases as if they're two different substances (they do behave very differently!)

$\boxed{\text{One Example: } ^4He}$

How come ^4He is a boson?

Nucleus has 2 protons ($\Leftrightarrow$ is Helium)
2 neutrons (4-2)

both protons & neutrons are (composite) spin-$\frac{1}{2}$ fermions.