Previously

We studied Fermi gases in the ultra-low-temperature régime: $T \ll \varepsilon_F$

$(\varepsilon_F \equiv \mu(T=0))$

Fermi energy $\varepsilon_F = \frac{\hbar^2}{2M}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$

$U(T=0) = \frac{3}{5} N \varepsilon_F$ ← this is *nonzero*!

Density of states – <u>available</u> orbitals –

$$\mathcal{D}(\varepsilon) \equiv \frac{dN}{d\varepsilon} = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$$

Density of occupied orbitals: $\mathcal{D}(\varepsilon)\, f(\varepsilon)$

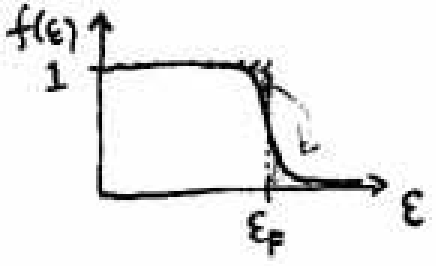
Fermi-Dirac distribution function

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/T} + 1} \xrightarrow[T \ll \varepsilon_F]{} \frac{1}{e^{(\varepsilon-\varepsilon_F)/T} + 1}$$

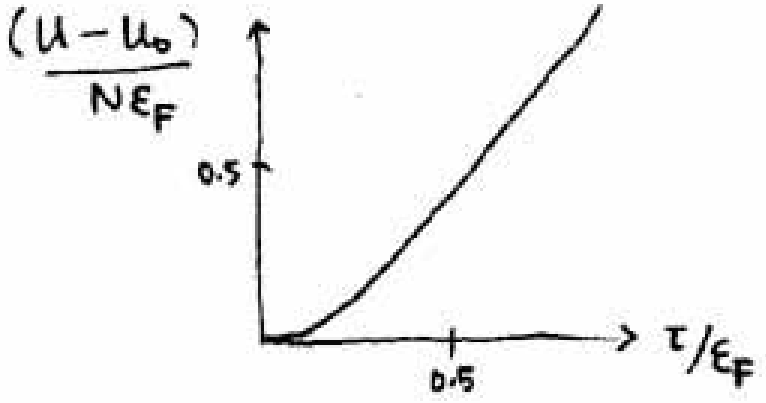
Heat capacity derived last lecture-

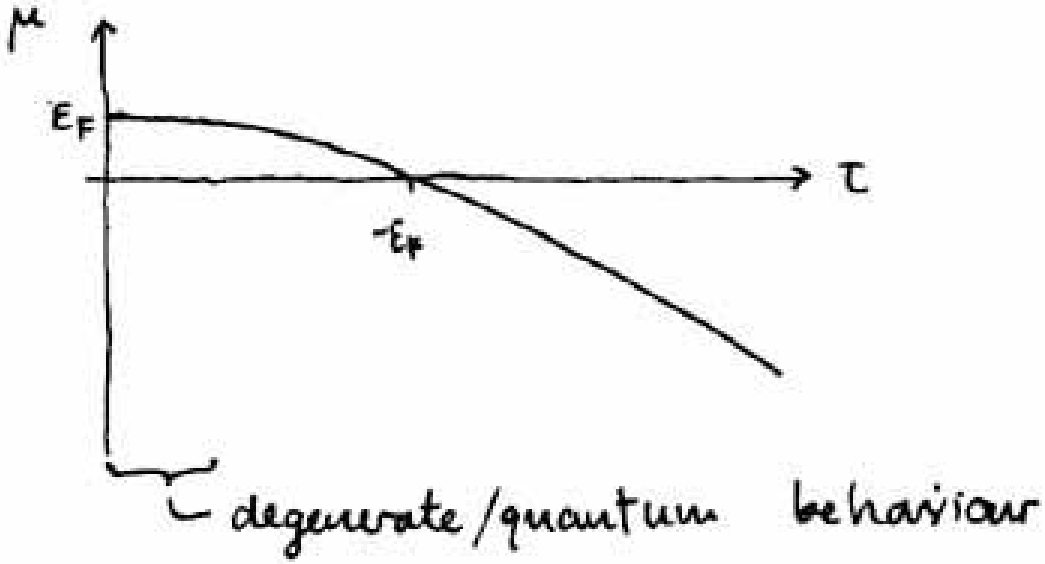
$$\boxed{C_{el} \underset{T \ll \varepsilon_F}{\approx} \frac{\pi^2}{2} N \left(\frac{T}{\varepsilon_F}\right)}$$

linear in T



Energy & Chemical Potential for Fermi Gas





Application: 'Gas' of conduction electrons in metals

Contribution to heat capacity from Fermi gas behaviour is linear in T.

Recall also Debye theory -

Debye story said that at low enough temperature, upper bound of $3N$ on # of phonon modes was not "felt"

So phonon gas acted like photon gas & had

$U \propto T^4 \Rightarrow C_V \propto T^3$

So for a metal, C_V has 2 contributions

(a) atomic lattice vibrations

$$C_V^{(at)} \equiv AT^3$$

(b) electron gas

$$C_V^{(el)} \equiv \gamma T$$

and $$\gamma = \frac{\pi^2}{2} N \frac{k_B}{\varepsilon_F}$$

while $$A = \frac{12\pi^4}{5} N \frac{1}{(k_B \Theta_D)^3}$$

Θ_D ↑ Debye temperature

Notice that by our usual definition

$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V$ C_V is <u>dimensionless</u>.

Let's look harder at these ⤴ formulae :

$$C_V = \gamma T + A T^3$$

$$= \frac{\pi^2}{2} N \left(\frac{T}{\varepsilon_F}\right) + \frac{12\pi^4}{5} N \left(\frac{T}{k_B \Theta_D}\right)^3$$

We can write $\varepsilon_F \equiv k_B T_F$

then

$$\left(\frac{C_V}{N}\right) = \left(\frac{\pi^2}{2}\right)\left(\frac{\varepsilon_F}{k_B T_F}\right) + \left(\frac{12\pi^4}{5}\right)\left(\frac{T}{\Theta_D}\right)^3$$

Both terms here are of the form

$$(\text{math \#'s}) \left(\frac{T}{T_{\text{characteristic}}}\right)^{\#}$$ ↑ integer power

This is another example of dimensional analysis at work in physics, and it's deep.

The N is there because C_V is extensive.

Characteristic temperatures?

$$T_F = \frac{1}{k_B}\varepsilon_F = \frac{1}{k_B}\left(\frac{\hbar^2}{2m}\right)\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

"Fermion Elbows"

Note that the elbows effect gives smaller T_F for

larger m

smaller concentrations $n = \frac{N}{V}$

and is proportional to $\hbar^2$

– if $\hbar$ were zero (no "quantum weirdness") then there'd be no effect ☺.

Regarding the Debye temperature

$$\Theta_D = \frac{\hbar v}{k_B} \left(6\pi^2 \frac{N}{V}\right)^{1/3}$$

The "mode capping" effect is what this embodies.

Mode capping ⇒ lower Θ_D

when v is smaller

(concentration (of atoms) is smaller

This characteristic temperature depends on ONE power of $\hbar$ (rather than 2 as for T_F) because we used a relativistic dispersion relation!

$k_B T_F = \left(\frac{\hbar^2}{2m}\right)\left(3\pi^2 \frac{N}{V}\right)^{\frac{2}{3}}$ ← electron concentration

c.f. $k_B \Theta_D = \hbar v \left(6\pi^2 \frac{N_a}{V}\right)^{1/3}$ ← atom concentration

Why does v appear for vibrational part?

$k_B \Theta_F$ dim's of energy; n_a has units of L^{-3}

so $\hbar (6\pi^2 n)^{\frac{1}{3}}$ has units of [energy][time][$\frac{1}{\text{length}}$]

so need $\left[\frac{\text{length}}{\text{time}}\right]$ to make correct dimensions

and that's what v, the speed of sound achieves.

Similarly, $(\cdots)^{2/3}$ has dimensions $\left[\frac{1}{(\text{length})^2}\right]$

$\left[\frac{\hbar^2}{2m}\right]$ = $\left[\frac{1}{\text{mass}}\right]([\text{momentum}][\text{length}])^2$

So RHS of $k_B T_F$ has dimensions

$\left[\frac{1}{\text{mass}} (\text{momentum})^2\right]$ like "$\frac{p^2}{2m}$" which is kinetic energy

So it works!

Let's work out T_D for a two-dimensional crystal!

In 3-d we had

$$k_B \theta_D = \hbar v \left(6\pi^2 \frac{N_a}{V}\right)^{\frac{1}{3}}$$

So let's suppose

$$k_B \theta_D^{(d=2)} = \hbar^p v^p \left(c \cdot \frac{N_a}{A}\right)^q \qquad p, q \text{ rational}$$

$\hbar$: energy of phonons $\propto \hbar$

RHS dimensions? $\left[\frac{N_a}{A}\right]^q = [\text{length}]^{-2q}$

LHS dim's of energy

$[\hbar] = [\text{energy} \cdot \text{time}]$ or $[\text{momentum} \cdot \text{length}]$

so $[\text{energy}] = [\text{energy} \cdot \text{time}] \left[\frac{\text{length}}{\text{time}}\right]^p [\text{length}]^{-2q}$

so we need $p = 1$ to cancel [time]

hence $q = \frac{1}{2} \Rightarrow k_B \theta_D = \hbar v \left(c \cdot \frac{N_a}{A}\right)^{\frac{1}{2}}$

The actual formula is

$$k_B \theta_D = \hbar v \left(2\pi \frac{N_a}{A}\right)^{\frac{1}{2}}$$

Experiments on metals?

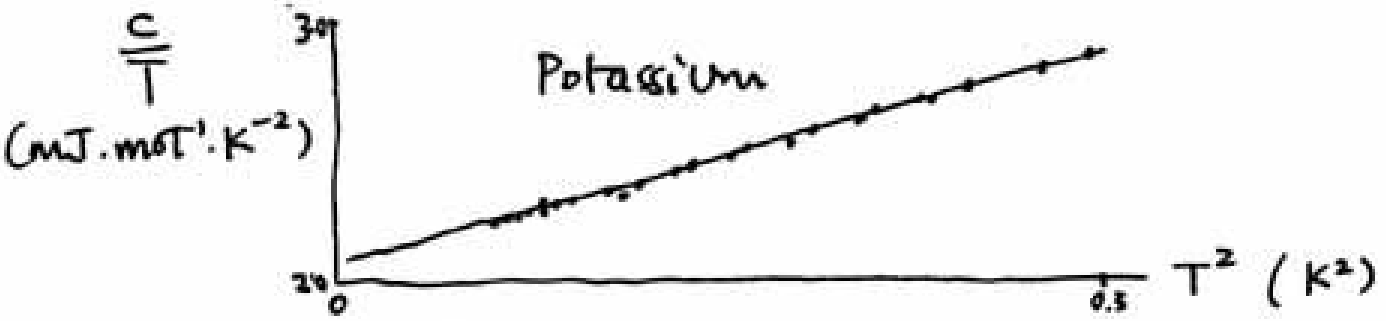
(1) E_F is order 1 eV for

Alkali metals (Li, Na, K, Rb, Cs)

and Copper, Silver, Gold

(2) Curves of $\frac{C_V}{T} \equiv \gamma + A T^2$ from theory

are, experimentally,



$\frac{C}{T} = 2.08 + 2.57\,T^2$ in these units