Then

$$N = \int_0^\infty d\varepsilon \; \mathcal{D}(\varepsilon) f(\varepsilon)$$

$$\langle \varepsilon \rangle = \int_0^\infty d\varepsilon \; \mathcal{D}(\varepsilon) f(\varepsilon) \, \varepsilon$$

For the ground state ONLY, f is a step function:

$$N_0 = \int_0^{\varepsilon_F} d\varepsilon \; \mathcal{D}(\varepsilon)$$

$$U_0 = \int_0^{\varepsilon_F} d\varepsilon \; \mathcal{D}(\varepsilon) \, \varepsilon$$

Heat Capacity

We know that our Fermi gas has energy U_0 even at absolute zero. We'd like to know what happens when we heat it up to temperature τ, and compute the heat capacity.

We don't want to include $\frac{\partial U_0}{\partial \tau}$ in our computation, because we're interested in only what it takes to bump up fermions above $\tau = 0$

Therefore, we want

$$C = \frac{\partial}{\partial \tau}\left\{ \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon) \cdot \varepsilon - \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon)\, \varepsilon \right\}$$

Now- who in $\{\ldots\}$ depends on τ?

Well, $\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2}$ doesn't

But $f(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} + 1}$ does.

It's annoying to have $\int_0^{\varepsilon_F}$ in the second term and not $\int_0^\infty \ldots$ so we need a trick.

We can use the fact that the # of particles is the same no matter τ:

$$N = \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon) = \int_0^\infty d\varepsilon\, f(\varepsilon)\, \mathcal{D}(\varepsilon)$$

so $\int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon)\, \varepsilon_F = \int_0^\infty d\varepsilon\, f(\varepsilon)\, \mathcal{D}(\varepsilon)\, \varepsilon_F$

i.e.

$C = \frac{\partial}{\partial \tau}\left\{ u(\tau) - u(0) \right\}$ where

$U(\tau) - U(0)$

$$= \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon)\, \varepsilon - \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon)\, \varepsilon + N\varepsilon_F - N\varepsilon_F$$

$$= \int_0^\infty d\varepsilon\, \mathcal{D}(\varepsilon) f(\varepsilon) (\varepsilon - \varepsilon_F) - \int_0^{\varepsilon_F} d\varepsilon\, \mathcal{D}(\varepsilon) (\varepsilon - \varepsilon_F)$$

$$= \int_{\varepsilon_F}^\infty d\varepsilon\, (\varepsilon - \varepsilon_F)\, f(\varepsilon)\, \mathcal{D}(\varepsilon) + \int_0^{\varepsilon_F} d\varepsilon\, (\varepsilon_F - \varepsilon)(1 - f(\varepsilon))\, \mathcal{D}(\varepsilon)$$

energy needed to take election from ε_F to $\varepsilon > \varepsilon_F$;
$f(\varepsilon)\, \mathcal{D}(\varepsilon)\, d\varepsilon$ = infinitesimal # elevated from ε_F to ε

$1 - f(\varepsilon)$ is probability e^- removed from orbital @ ε

So !

$$C_{el} = \frac{\partial}{\partial \tau}\left(U(\tau) - U(0)\right) = \int_{\varepsilon_F}^\infty d\varepsilon\, (\varepsilon - \varepsilon_F) \frac{\partial f}{\partial \tau} \mathcal{D}(\varepsilon) + \int_0^{\varepsilon_F} d\varepsilon\, (\varepsilon_F - \varepsilon) \cdot \frac{-\partial f}{\partial \tau} \mathcal{D}(\varepsilon)$$

In other words,

$$C_{el} = \int_0^\infty d\varepsilon \, (\varepsilon - \varepsilon_F) \frac{\partial f}{\partial \tau} \mathcal{D}(\varepsilon)$$

Now: $\dfrac{\partial}{\partial \tau} \dfrac{1}{e^{(\varepsilon-\mu)/\tau} + 1} = \; ?$

(For $\tau \ll \varepsilon_F$ ignore temperature dependence on μ says book.) & Replace μ by ε_F

$$\frac{\partial f}{\partial \tau} = -\frac{(\varepsilon-\varepsilon_F)}{\tau^2} \cdot \frac{-1}{(e^{(\varepsilon-\cancel{\mu}\,\varepsilon_F)/\tau} + 1)^2} \cdot e^{(\varepsilon-\varepsilon_F)/\tau} .$$

Now let $x \equiv \dfrac{(\varepsilon - \varepsilon_F)}{\tau}$.

$$\Rightarrow \quad C_{el} = \tau \int_{-\varepsilon_F/\tau}^{\infty} dx \, \frac{x^2 e^x}{(e^x+1)^2} \mathcal{D}(\varepsilon)$$

what will we do with this?

For $\varepsilon_F \gg \tau$, f only has a noticeable derivative near $\varepsilon = \varepsilon_F$. $\Rightarrow$ approximate as $\mathcal{D}(\underline{\underline{\varepsilon_F}})$.

Also for $\tau \ll \varepsilon_F$, lower limit $\sim -\infty$

$$\Rightarrow \quad C_{el} \simeq \tau \mathcal{D}(\varepsilon_F) \int_{\infty}^{+\infty} dx \frac{x^2 e^x}{(e^x+1)^2} \Rightarrow \boxed{C_{el} \cong \frac{\pi^2}{3} \tau \mathcal{D}(\varepsilon_F)}$$

We had

$$\mathcal{D}(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} \qquad \text{and}$$

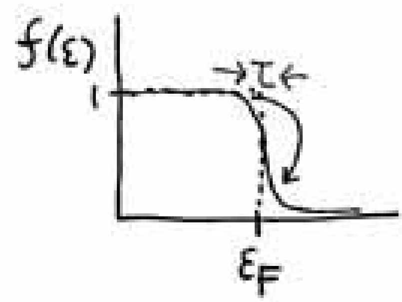
$$N = \frac{V}{3\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \varepsilon_F^{3/2}.$$

so $\quad \mathcal{D}(\varepsilon_F)\cdot\varepsilon_F = \frac{3}{2}N$

so

$$\boxed{C_{el} \underset{\tau \ll \varepsilon_F}{\cong} N \frac{\pi^2}{2}\frac{\tau}{\varepsilon_F}}$$

Linear in τ



Why did this happen?

excited e^-s for low-τ

is $\sim N \frac{\tau}{\varepsilon_F}$

and energy increase $\sim \tau$

so $\quad U(\tau) - U(0) \sim \frac{N\tau^2}{\varepsilon_F}$

so $\quad C_{el} \sim N\left(\frac{\tau}{\varepsilon_F}\right)$.