PHY484S/1484S GR1 (2018-19) – HW4 – due @ 10am M01Apr2019

4.1: Stability of planar Newtonian orbits in higher dimensions

Consider the motion of a planet in a planar circular orbit around a heavy star in dimension $D \geq 4$. Keeping the motion planar, consider a small perturbation of the circular orbit, like might occur if a meteorite moving in the orbit plane hit the planet.

(a) Set up the problem by using $\vec{x} = r(t)\hat{e}_r(t)$ for the planet of mass m. By recruiting conservation of angular momentum L for the planet in the central gravitational force described by the Newtonian potential $\Phi_G^{(D)}$, show that

$$\ddot{r} = -\frac{\partial}{\partial r} V_{\rm eff}(r)\,, \qquad V_{\rm eff}(r) \equiv \frac{\Phi_G^{(D)}}{m} + \frac{L^2}{2m^2 r^2}\,. \tag{1}$$

(b) Using your result from part (a) and

$$\Phi_G^{(D)} \equiv -\frac{\alpha_D G_D M}{r^{D-3}}\,, \qquad \alpha_D = \text{const.}\,, \tag{2}$$

show that while planar circular orbits for planets are stable under small planar perturbations in $D = 4$, they are not stable in $D = 5$ or in $D > 5$.

4.2: Newtonian potentials in 5D vs 4D

(a) Consider 5D in Cartesian coordinates $\{t, x, y, z, w\}$ and put a point mass M at the origin. By starting from the Poisson equation $\vec{\nabla}^2 \Phi_G^{(5)} = 4\pi G_5 \rho_M$ and integrating, show that

$$\Phi_G^{(5)} = -\frac{G_5 M}{\pi \hat{r}^2}\,, \qquad \hat{r}^2 = x^2 + y^2 + z^2 + w^2 \equiv r^2 + w^2\,. \tag{3}$$

(b) What happens if we want to roll up w on a circle of radius R while keeping M fixed? This requires a periodic potential. The slickest way to find the answer is to realize that if we keep w unrolled, we need an infinite set of image charges located $2\pi R$ apart along the w axis. Write down the Newtonian potential for this setup. What is $\Phi_G^{(5)}(x, y, z, 0)$ in the limit that $r \gg R$? You should recover the familiar 4D Newtonian potential with $G_4 = G_5/(2\pi R)$.

(c) A handy identity is

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + (\pi n u)^2} = \frac{1}{u} \coth\left(\frac{1}{u}\right)\,. \tag{4}$$

Use this to find (i) the leading correction to the 4D Newtonian potential for $r \gg R$; and (ii) the leading behaviour of $\Phi_G^{(5)}$ when $r \ll R$.

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4.3: Hawking temperature of Tagherlini black holes

When we study QM, we obsess about $e^{-iH\Delta t}$. When we study thermal physics, we obsess about $e^{-\beta H}$, where $\beta = 1/T$. These two setups are formally related to each other by Euclidean continuation, $t_E = it$. Specifically[1], the *periodicity in Euclidean time plays the role of the inverse temperature*. Here we will not explain this neat fact but simply recruit it to find the Hawking temperature of a class of black holes in the most painless way possible.

Consider the spacetime metric for an asymptotically flat neutral black hole in $D \geq 4$,

$$ds^2 = \left(1 - \frac{r_H^{D-3}}{r^{D-3}}\right) dt^2 - \left(1 - \frac{r_H^{D-3}}{r^{D-3}}\right)^{-1} dr^2 - r^2 d\Omega_{D-2}^2 \,, \tag{5}$$

where the horizon radius r_H is related to the mass M via

$$r_H^{D-3} = \frac{16\pi G_D M}{(D-2)\Omega_{D-2}} \,. \tag{6}$$

This was discovered by Tagherlini[2] in 1963 and is the higher-D version of Schwarzschild.

(a) Check using Maxima that this metric satisfies the vacuum Einstein equation with zero cosmological constant – just for the 5D case.

(b) Show that, near the horizon (and outside it),

$$ds^2 \simeq \frac{\eta^2}{4r_H^2} dt^2 - d\eta^2 - r_H^2 d\Omega_{D-2}^2 \,, \tag{7}$$

where η is the proper distance and $D \geq 4$.

(c) Now continue the time coordinate to Euclidean time, and do a simple rescaling to make the first two terms of the line element look very familiar. Explain geometrically why the rescaled Euclidean time must be periodically identified with period 2π. Undoing the rescaling, show that this gives the Hawking temperature

$$T_H = \frac{1}{4\pi r_H} \,. \tag{8}$$

Use this to explain why the heat capacity of Tagherlini black holes is negative.

(d) To estimate how fast a black hole loses mass, model it as a blackbody with the area of the horizon at the Hawking temperature and recruit the Stefan-Boltzmann law in D dimensions to find dM/dt. Show that the lifetime of the black hole scales as

$$\Delta t \sim G_D^{2/(D-3)} M^{(D-1)/(D-3)} \,. \tag{9}$$

For $D = 4$, show that a primordial black hole with a lifetime of the age of the universe began life with a mass of roughly 10^{12}kg. How big was its Schwarzschild radius in metres?

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[1] Tony Zee has a nice explanation of this in his "QFT in a Nutshell" textbook, §V.2.
[2] Why $D \geq 4$? In $D \leq 3$, having a point mass is not compatible with asymptotic flatness.

4.4: Making a new charged KK black hole in 4D

(a) Consider a neutral black string spacetime in 5D with coordinates $\{\hat{t}, r, \theta, \phi, \hat{z}\}$,

$$d\hat{s}^2 = f_h(r)d\hat{t}^2 - \frac{1}{f_h(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) - d\hat{z}^2\,, \qquad f_h(r) \equiv \left(1 - \frac{r_H}{r}\right)\,. \tag{10}$$

Why does this spacetime solve the Einstein equations in 5D?

(b) Now do a Lorentz boost on this black string with rapidity ζ,

$$\begin{aligned} d\hat{t} &= \cosh\zeta\, dt + \sinh\zeta\, dz\,, \\ d\hat{z} &= \sinh\zeta\, dt + \cosh\zeta\, dz\,. \end{aligned} \tag{11}$$

Next, reduce the boosted 5D black string down to 4D along the z direction (*not* along the $\hat{z}$ direction!), using the Kaluza-Klein reduction formula from lecture notes,

$$d\hat{s}^2 = e^{2\alpha\chi}ds^2 - e^{2\beta\chi}\left(dz + A_\mu dx^\mu\right)^2\,, \quad \beta = -(D-2)\alpha\,, \quad \alpha^2 = \frac{1}{2(D-1)(D-2)}\,. \tag{12}$$

You should obtain

$$\begin{aligned} e^{2\beta\chi} &= 1 + \frac{r_H \sinh^2\zeta}{r} \equiv f_s(r)\,, \qquad \beta = \frac{\mp 1}{\sqrt{3}}\,, \\ A_t &= \frac{q_e}{r} f_s^{-1}(r)\,, \qquad q_e \equiv r_H \cosh\zeta \sinh\zeta\,, \\ ds^2 &= \frac{f_h(r)}{\sqrt{f_s(r)}}dt^2 - \sqrt{f_s(r)}\left[\frac{dr^2}{f_h(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]\,. \end{aligned} \tag{13}$$

Hint: the smart way to find the 4D metric, vector, and scalar from the 5D metric in eq.(12) is to extract $e^{2\beta\chi}$ first from $\hat{g}_{zz}$, then A_μ from $\hat{g}_{z\mu}$, and lastly $g_{\mu\nu}$ from $\hat{g}_{\mu\nu}$.

(c) Starting from our Kaluza-Klein action

$$S[\chi, A_\sigma, g_{\lambda\rho}] = \frac{1}{16\pi G_D}\int d^Dx\,\sqrt{-g}\left[R_g + \frac{1}{2}(\partial\chi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\chi}F^2\right]\,, \tag{14}$$

show that the field equations for our scalar, vector, and spacetime metric in 4D are

$$\begin{aligned} \nabla^\mu\nabla_\mu\chi &= -\frac{3\beta}{4}e^{3\beta\chi}F^2\,, \\ \nabla_\mu\left(e^{3\beta\chi}F^{\mu\nu}\right) &= 0\,, \\ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= -\frac{1}{2}T_{\mu\nu}\,, \\ T_{\mu\nu} &= \nabla_\mu\chi\nabla_\nu\chi - \frac{1}{2}g_{\mu\nu}\left(\nabla\chi\right)^2 + e^{3\beta\chi}\left(-F_{\mu\sigma}F^\sigma{}_\nu + \frac{1}{4}g_{\mu\nu}F^2\right)\,. \end{aligned} \tag{15}$$

Show that your new solution (13) does indeed solve all three field equations. You will probably want to use Maxima to assist, at least for finding the 4D Einstein tensor.