PHY484S/1484S GR2 (2020-21) – HW3 – due @ 10am M22Mar

3.1: Geodesics from particle energy-momentum conservation *[25]*

The principle of covariant conservation of energy-momentum can be used to derive the geodesic equation for a point particle providing the source of energy-momentum.

(a) *[7]* Starting from the Einbein action for a point particle coupled to gravity,

$$S_{\rm einbein} = \int d\lambda \frac{1}{2}\left[e^{-1}(\lambda)\dot{z}^2(\lambda) + e(\lambda)m^2\right] , \tag{1}$$

and the definitions

$$S_{\rm matter} \equiv \int d^D x\, \mathscr{L}_{\rm matter}\,, \quad T_{\alpha\beta} \equiv \frac{2}{\sqrt{-g}}\frac{\delta \mathscr{L}_{\rm matter}}{\delta g^{\alpha\beta}}\,, \tag{2}$$

show that

$$T_{\mu\nu}(x) = \frac{m}{\sqrt{-g(x)}}\int d\lambda\, \dot{z}_\mu(\lambda)\dot{z}_\nu(\lambda)\, \delta^{(D)}(x - z(\lambda))\,. \tag{3}$$

where in the massless case we really mean 1 instead of m, and $\dot{} \equiv d/d\lambda$.
Hint: you may fix the einbein via $e^{-1}(\lambda) = m$ (massive, 'proper time gauge') or $e^{-1}(\lambda) = 1$ (massless), where λ is the affine parameter for the particle path $z^\mu(\lambda)$.

(b) *[9]* Show that for an arbitrary rank (2,0) tensor T

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu\nu}\right) + \Gamma^\nu{}_{\sigma\mu}T^{\mu\sigma}\,. \tag{4}$$

(c) *[9]* Where does the equation expressing covariant conservation of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0 \tag{5}$$

come from? Use this and results from (a) and (b), along with an integration by parts, to show that covariant conservation of the particle energy-momentum tensor requires

$$\nabla_\mu T^{\mu\nu} = \int d\lambda \left[\frac{d^2 z^\nu(\lambda)}{d\lambda^2} + \Gamma^\nu{}_{\alpha\beta}(z(\lambda))\frac{dz^\alpha(\lambda)}{d\lambda}\frac{dz^\beta(\lambda)}{d\lambda}\right]\delta^D(x - z(\lambda)) = 0\,. \tag{6}$$

Where have you seen the piece inside square brackets before?

3.2: Explaining *[25]*

Describe the key concepts involved in cosmological perturbations, for an audience of students who have finished the first GR course (PHY483F/1483F) but are not taking this course.

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3.3: Scalar field perturbations *[20]*

Suppose that our scalar inflaton φ with potential $V(\varphi)$ is written as a homogeneous isotropic part plus perturbations:

$$\varphi(t,\vec{x}) = \varphi_0(t) + \delta\varphi(t,\vec{x})\,. \tag{7}$$

Show that, to first order in small quantities, the perturbed rank (1,1) energy-momentum tensor obeys

$$\begin{aligned}
\delta T^0{}_i &= \dot{\varphi}_0\,\partial_i(\delta\varphi)\,,\\
\delta T^0{}_0 &= -\dot{\varphi}_0^2\,\Phi + \dot{\varphi}_0\,\delta\dot{\varphi} + V'\delta\varphi\,,\\
\delta T^i{}_j &= \delta^i_j\left(\dot{\varphi}_0^2\,\Phi - \dot{\varphi}_0\,\delta\dot{\varphi} + V'\delta\varphi\right)\,.
\end{aligned} \tag{8}$$

where $' \equiv d/d\varphi$ and Φ is the scalar metric perturbation in Newtonian gauge given below in (9). Are the components $\delta T^i{}_0$ independent of those listed in (8)?

3.4: Metric perturbations *[30]*

Suppose that spacetime perturbations about a $k = 0$ FRW universe can be described in Newtonian gauge by

$$ds^2 = [1 + 2\Phi(t,\vec{x})]\,dt^2 - [1 - 2\Phi(t,\vec{x})]\,a^2(t)|d\vec{x}|^2\,. \tag{9}$$

Show that, to first order in Φ, the perturbed Christoffels are

$$\begin{aligned}
\delta\Gamma^0{}_{0\mu} &= \partial_\mu\Phi\,,\\
\delta\Gamma^0{}_{ij} &= -\delta_{ij}a^2\left(\dot{\Phi} + 4H\Phi\right)\,,\\
\delta\Gamma^i{}_{00} &= a^{-2}\delta^{ij}\partial_j\Phi\,,\\
\delta\Gamma^i_{0j} &= -\delta^i_j\dot{\Phi}\,,\\
\delta\Gamma^i{}_{jk} &= -\delta^i_k\partial_j\Phi - \delta^i_j\partial_k\Phi + \delta_{jk}\delta^{i\ell}\partial_\ell\Phi\,,
\end{aligned} \tag{10}$$

and others related to these by symmetry. Use these results to show that, again to first order in small quantities, the perturbed rank (1,1) Einstein tensor obeys

$$\begin{aligned}
\delta G^0{}_i &= -2\partial_i\left(\dot{\Phi} + H\Phi\right)\,,\\
\delta G^0{}_0 &= -2\left(\vec{\nabla}^2\Phi - 3H\dot{\Phi} - 3H^2\Phi\right)\,,\\
\delta G^i{}_j &= 2\delta^i_j\left[\ddot{\Phi} + 4\dot{\Phi}H + \left(2\dot{H} + 3H^2\right)\Phi\right]\,.
\end{aligned} \tag{11}$$

Are the components $\delta G^i{}_0$ independent of those listed in (11)?
