PHY484S/1484S GR1 (2018-19) – HW1 – due @ 10am M04Feb2019

1.2: Kasner cosmology

(a) Explain why the Einstein equations can be rewritten as

$$R^\mu{}_\nu = -8\pi G_N \left[T^\mu{}_\nu - \frac{1}{(D-2)} T \delta^\mu{}_\nu \right] + \frac{2}{(D-2)} \delta^\mu{}_\nu \Lambda \,. \tag{8}$$

For the rest of this problem, assume that spacetime takes the Kasner form,

$$ds^2 = dt^2 - \sum_{i=1}^{3} [a_i(t)]^2 (dx^i)^2 \,. \tag{9}$$

Note that the spatial sections of this metric are flat. Its key physical feature is its anisotropy: different directions in spacetime have *different scale factors*.

(b) Show by hand that the Ricci tensor $R^\mu{}_\nu$ of this Kasner spacetime has components

$$R^0{}_0 = \sum_{i=1}^{3} \frac{\ddot{a}_i}{a_i} \,, \qquad R^1{}_1 = \frac{\ddot{a}_1}{a_1} - \left(\frac{\dot{a}_1}{a_1}\right)^2 + \left(\sum_{k=1}^{3} \frac{\dot{a}_k}{a_k}\right) \frac{\dot{a}_1}{a_1} \,, \tag{10}$$

and similarly for $R^2{}_2, R^3{}_3$. Hint: write out all your summations over spatial indices explicitly, even those you wouldn't normally bother showing, in order to avoid errors.

(c) Show that if you try *power-law* scale factors

$$a_i(t) = t^{q_i} \tag{11}$$

in the *vacuum* Einstein equations with $\Lambda = 0$, you get

$$\sum_{i=1}^{3} q_i^2 = 1 \,, \qquad \sum_{i=1}^{3} q_i = 1 \,. \tag{12}$$

Argue from this that isotropic power-law evolution is forbidden in vacuum for $\Lambda = 0$.

(d) Now allow a $T^\mu{}_\nu$ in the Einstein equations and try *isotropic, exponential* expansion,

$$a_i(t) = e^{Ht} \quad \forall i \,. \tag{13}$$

Are perfect fluid solutions with constant energy density and pressure allowed? What do ρ and p have to be? Explain. Start your answer by showing that (i) covariant conservation of energy-momentum $\nabla_\mu T^\mu{}_\nu = 0$ in the anisotropic Kasner spacetime requires that the perfect fluid satisfies

$$\frac{\partial \rho}{\partial t} + \left(\sum_{k=1}^{3} \frac{\dot{a}_k}{a_k}\right)(\rho + p) = 0 \,, \tag{14}$$

and that (ii) the isotropic exponentially expanding scale factor $a(t)$ must obey the Einstein equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G_N}{3} \rho \,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3} (\rho + 3p) \,. \tag{15}$$