SAMPLE PAST MIDTERM SNIPPETS FOR PRACTISING ON

Concepts

- What is the defining property of a tensor? Give four examples of physically useful tensors in the context of Special Relativity.
- In curved spacetime, why is the partial derivative of a tensor not a tensor? How do we build a covariant derivative of a tensor that *is* a tensor?
- Why does a geodesic, the path followed by a freely falling observer in curved spacetime, *maximize* the proper time?
- Explain the physical significance of the geodesic deviation equation.
- Sketch how taking the Newtonian limit of the geodesic deviation equation in curved spacetime gives back the line deviation equation pertinent to tidal forces, working at linearized order in small quantities.

Calculations: index gymnastics

A conformal transformation takes one metric $g_{\mu\nu}$ to another $\tilde{g}_{\mu\nu}$ described by

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \,, \tag{1}$$

where $\Omega(x)$ is a scalar function of spacetime coordinates known as the conformal factor.

(a) What distinguishes a null particle trajectory from a timelike one? Explain why the trajectories of light beams must remain unchanged under conformal transformations of the form (1).

(b) Starting from the definition of the Christoffels in terms of the metric, prove that the Christoffels $\tilde{\Gamma}^{\rho}{}_{\mu\nu}$ for the conformally transformed metric are related to the original ones $\Gamma^{\rho}{}_{\mu\nu}$ by

$$\tilde{\Gamma}^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} + \Omega^{-1}\left(\delta^{\sigma}{}_{\mu}\nabla_{\nu}\Omega + \delta^{\sigma}{}_{\nu}\nabla_{\mu}\Omega - g_{\mu\nu}g^{\sigma\lambda}\nabla_{\lambda}\Omega\right) \,. \tag{2}$$

Note that the difference between the two Christoffels is a bona fide tensor, as it must be.

Calculations: Christoffels, Riemann, and geodesics

Consider AdS_3 in global coordinates,

$$ds^2 = L^2\left[-\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, d\phi^2\right] \,. \tag{3}$$

(a) Use this metric in eq.(3) to show that the nonzero Christoffels are

$$\Gamma^{t}{}_{tr}(=\Gamma^{t}{}_{rt}) = \frac{\sinh r}{\cosh r} \,, \qquad \Gamma^{r}{}_{tt} = \sinh r \cosh r \,, \qquad \Gamma^{\phi}{}_{\phi r}(=\Gamma^{\phi}{}_{r\phi}) \,, \qquad \Gamma^{r}{}_{\phi\phi} \,. \tag{4}$$

Focus on finding $\Gamma^{t}{}_{tr}$ and $\Gamma^{r}{}_{tt}$. The precise functional form of $\Gamma^{\phi}{}_{\phi r}$ and $\Gamma^{r}{}_{\phi\phi}$ will not matter for most of the rest of this question.

(b) Find $R^{t}{}_{rtr}$ from the Christoffels above. After raising its second index, you should obtain

$$R^{tr}{}_{tr} = +\frac{1}{L^2} \,. \tag{5}$$

If you have time to spare, work out the remaining two Christoffels in eq.(4) and show also that

$$R^{t\phi}{}_{t\phi} = +\frac{1}{L^2} \,, \quad R^{r\phi}{}_{r\phi} = +\frac{1}{L^2} \,. \tag{6}$$

(c) Using the Christoffels from eq.(4), show that the geodesic equations for radial motion are

$$\frac{d^2t}{d\lambda^2} + 2\frac{dt}{d\lambda}\frac{dr}{d\lambda}\frac{\sinh r}{\cosh r} = 0\,, \tag{7}$$

$$\frac{d^2r}{d\lambda^2} + \left(\frac{dt}{d\lambda}\right)^2 \sinh r \cosh r = 0\,. \tag{8}$$

Show that the first of these two equations eq.(7) has a first integral yielding a conservation law,

$$\mathcal{E} = \cosh^2 r\,\frac{dt}{d\lambda} = \text{const.} \tag{9}$$

(d) A null geodesic has a null tangent vector,

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0\,. \tag{10}$$

Assuming radial motion, use this nullness condition eq.(10) and the conservation law eq.(9) to find an equation for $dr/d\lambda$ and integrate it to get $r(\lambda)$. Then use your solution for $r(\lambda)$ to find $dt/d\lambda$ and integrate it to get $t(\lambda)$. Does it take a finite or infinite amount of coordinate time t to reach $r \to \infty$, starting from $r = 0$? Handy integrals (c, c' are constants):

$$\int dx\,\cosh x = \sinh x + c \qquad\qquad \int \frac{dx}{1+x^2} = \arctan(x) + c' \tag{11}$$

If you can do all of the above, try doing the question again but with a different 3D spacetime, or with the same spacetime but in a different coordinate system. If you fly through that too and want an extra challenge, try finding Killing vectors of the above AdS_3 metric.

Formula sheet

Electric and magnetic fields in $D = 3+1$ flat Minkowski spacetime with Cartesian coordinates:-

$$F_{0i} = -\delta_{ij} E^j \,, \; F_{ij} = \mathfrak{E}_{ijk} B^k \,, \text{ where } \mathfrak{E}_{ijk} = +1(-1) \text{ if } ijk \text{ even (odd) perm. of 123 or 0 otherwise} \tag{12}$$

Invariant interval:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{13}$$

Coordinate transformation law for rank (m,n) tensors V:

$$V^{\mu_1{}' \ldots \mu_m{}'}{}_{\nu_1{}' \ldots \nu_n{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\lambda_1}} \cdots \frac{\partial x^{\mu_m{}'}}{\partial x^{\lambda_m}} \frac{\partial x^{\sigma_1}}{\partial x^{\nu_1{}'}} \cdots \frac{\partial x^{\sigma_n}}{\partial x^{\nu_n{}'}} \; V^{\lambda_1 \ldots \lambda_m}{}_{\sigma_1 \ldots \sigma_n} \tag{14}$$

Indices are lowered with $g_{\mu\nu}$ and raised with $g^{\mu\nu}$, where $g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu$ and $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$, e.g. for vector V:

$$V_\mu = g_{\mu\nu} V^\nu \tag{15}$$

Downstairs covariant derivative of rank (m,n) tensor V:

$$\begin{aligned} \nabla_\sigma V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = \partial_\sigma V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} + \Gamma^{\mu_1}{}_{\sigma\lambda} V^{\lambda \mu_2 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} + \Gamma^{\mu_2}{}_{\sigma\lambda} V^{\mu_1 \lambda \mu_3 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} + \ldots \\ - \Gamma^{\lambda}{}_{\sigma\nu_1} V^{\mu_1 \ldots \mu_m}{}_{\lambda \nu_2 \ldots \nu_n} - \Gamma^{\lambda}{}_{\sigma\nu_2} V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \lambda \nu_3 \ldots \nu_n} + \ldots \end{aligned} \tag{16}$$

Directional covariant derivative along curve $x^\mu(\lambda)$:

$$\frac{D}{D\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu \tag{17}$$

Christoffel symbols, which are symmetric in $\nu \leftrightarrow \sigma$:

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\nu\sigma} \right) \tag{18}$$

Metric compatibility condition for Christoffel connection:

$$\nabla_\sigma g_{\mu\nu} = 0 \tag{19}$$

Geodesic equation for $x^\mu(\lambda)$, where λ is affine parameter:

$$\frac{D}{D\lambda} \left(\frac{dx^\mu}{d\lambda} \right) = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \tag{20}$$

Alternative form of geodesic equation:

$$\frac{d}{d\lambda} \frac{dx_\mu}{d\lambda} = \frac{1}{2} (\partial_\mu g_{\nu\sigma}) \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} \tag{21}$$

Riemann tensor:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda} \Gamma^\lambda{}_{\mu\sigma} \tag{22}$$

Riemann from commutator of covariant derivatives on a rank (k,ℓ) tensor T:

$$\begin{aligned} \left[\nabla_\rho, \nabla_\sigma \right] T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} = +R^{\mu_1}{}_{\lambda\rho\sigma} T^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} + R^{\mu_2}{}_{\lambda\rho\sigma} T^{\mu_1 \lambda \mu_3 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} + \ldots \\ - R^{\lambda}{}_{\nu_1\rho\sigma} T^{\mu_1 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_\ell} - R^{\lambda}{}_{\nu_2\rho\sigma} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \lambda \nu_3 \ldots \nu_\ell} - \ldots \end{aligned} \tag{23}$$

Symmetries of Riemann:

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \,, \qquad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \,, \qquad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \,, \qquad R_{[\alpha\beta\gamma\delta]} = 0 \tag{24}$$

Geodesic deviation equation, where S is separation vector and T is tangent vector:

$$\frac{D^2 S^\mu}{D\lambda^2} = (\nabla_T \nabla_T S)^\mu = -R^\mu{}_{\nu\alpha\sigma} T^\nu T^\alpha S^\sigma \tag{25}$$

Newtonian limit and line deviation equation ($x^0 = ct$):

$$\left|\frac{\vec{v}}{c}\right| \ll 1\,,\ |\partial_0| \ll |\partial_i|\ ,\ ds^2 \simeq \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) |d\vec{x}|^2\,,\ \left|\frac{\Phi}{c^2}\right| \ll 1$$
$$\frac{d^2}{dt^2} y^i = -\delta^{ij} (\partial_j \partial_k \Phi) y^k \tag{26}$$
