PHY483F/1483F (2018-19) – HW2 – due 11am R18Oct *[25 marks]*

2.1: Riemann computations for AdS4 *[10 marks]*

Consider the 4D spacetime metric

$$ds^2 = L^2\left[\cosh^2 r\, dt^2 - dr^2 - \sinh^2 r\left(d\theta^2 + \sin^2\!\theta\, d\phi^2\right)\right] , \tag{1}$$

where L is a constant.

(a)*[4]* Find the nonzero components of the Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ for this metric. To do this, first program Maxima to compute the Christoffels for you. Next, check your Maxima results by computing at least two representative nonzero Christoffels by hand. Then compute two representative nonzero components of Riemann by hand, to show that you understand how. Finally, program Maxima to automate the computation of the rest of the components of Riemann.
(N.B.: Maxima may define Riemann differently than we do. Check the `ctensor` documentation for their sign convention on defining `riem` *in terms of* `mcs`*. Don't forget to properly document your code.)*

(b)*[4]* Find the nonzero components of the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$ for this metric, by hand, from the Riemann components you found in (a). Then find the Ricci scalar $R = R^\mu{}_\mu$, again by hand.

(c)*[2]* Using your results from (b) and (c), show that this spacetime obeys the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0\,, \tag{2}$$

where Λ is a constant, and find the value of Λ in terms of L. You should find that $R \propto \Lambda \propto -1/L^2$. This spacetime is known as 4D Anti de Sitter (AdS). Λ is known as the cosmological constant.

2.2: Charged particle in curved spacetime *[6 marks]*

For a charged massive particle moving in curved spacetime, the free relativistic particle action needs to be supplemented by an extra term representing its interaction with the electromagnetic field. Up to a classically irrelevant constant, the combined action in proper time gauge is

$$S = \int d\tau\left[\frac{1}{2}m\, g_{\mu\nu}(x)\,\dot{x}^\mu(\tau)\,\dot{x}^\nu(\tau) + q\,A_\mu(x)\,\dot{x}^\mu(\tau)\right] = \int d\tau\, L\left(x^\mu, \dot{x}^\mu\right) , \tag{3}$$

where m is the mass and q is the electric charge.

(a)*[6]* Find the equations of motion for this system by starting from the above Lagrangian and using the Euler-Lagrange equations,

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\right] . \tag{4}$$

Be sure to remember that both the metric tensor $g_{\mu\nu}(x^\alpha)$ and the 4-vector potential $A_\mu(x^\alpha)$ depend on all the x^α coordinates, which in turn depend on τ for the moving particle. You should obtain

$$\frac{d^2x^\mu(\tau)}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}(x)\frac{dx^\nu(\tau)}{d\tau}\frac{dx^\sigma(\tau)}{d\tau} = \frac{q}{m}F^\mu{}_\nu(x)\frac{dx^\nu(\tau)}{d\tau} . \tag{5}$$

The left hand side should look familiar. The right hand side is nonzero because the charged particle feels an external force from the electromagnetic field. Accordingly, the relativistic particle path described by $x^\mu(\tau)$ is not a geodesic.

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2.3: GPS satellites and time dilation *[9 marks]*

(a) *[3]* Consider flat 4D Minkowski spacetime, written in (unprimed) spherical polar coordinates t, r, θ, ϕ. Now transform into a rotating (primed) frame of reference with

$$t' = t\,, \quad r' = r\,, \quad \theta' = \theta\,, \quad \phi' = \phi + \omega t\,. \tag{6}$$

Find the spacetime metric in the rotating frame of reference. Use it to find an expression for $d\tau/dt'$ in the rotating frame, where the infinitesimal proper time is $d\tau = ds/c$. Making the approximation that $|\omega r'/c| \ll 1$, show that for a clock stationary in the rotating frame

$$\frac{d\tau}{dt'} \simeq 1 - \frac{\omega^2 (r')^2 \sin^2\theta'}{2c^2}\,. \tag{7}$$

Therefore, if you synchronize clocks in an inertial reference frame, you cannot also synchronize them in a rotating reference frame. Rotating reference frames are non-inertial, light beams do not move in straight lines, and the total path length of a light beam will typically depend on the rotation (this is also known as the Sagnac effect). Life is much simpler in the original inertial frame of reference, which is why in discussions of GPS satellite timing, sensible physicists usually use an Earth-centred inertial (ECI) frame of reference.

For the rest of this question, stick with an ECI frame and ignore Earth's rotation about its own axis. For simplicity, you may also ignore the fact that Earth has a quadrupole (its shape is actually not round, but more like a squashed sphere). Finally, at our level of approximation, you may also neglect Earth's rotation around the Sun and the gravitational effect of other solar system bodies like the Moon.

(b) *[3]* Newtonian physics is recovered as a limit of GR where speeds are low and gravitational fields are weak. Assume that the approximate spacetime metric is

$$ds^2 \simeq \left(1 + \frac{2\Phi}{c^2}\right)(cdt)^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)\,, \tag{8}$$

where Φ is the Newtonian potential and Φ/c^2 is small compared to 1. Starting from the metric (8), show that, to first order in small quantities,

$$\frac{d\tau}{dt} \simeq 1 + \frac{\Phi}{c^2} - \frac{|\vec{v}|^2}{2c^2}\,. \tag{9}$$

This encapsulates both the general relativistic time dilation from the Earth's gravitational field and the special relativistic time dilation from the fact that the satellite travels at nonzero speed.

(c) *[3]* The Virial Theorem relates the average kinetic energy for a stable system to the average potential energy. For a $V \propto 1/r$ potential, when the system has two particles and is non-relativistic, it says that $2\langle T\rangle = -1\langle V\rangle$, where T is the kinetic energy and V is the potential energy. Apply the non-relativistic virial theorem to a satellite in circular orbit at radius r from the centre of Earth, and integrate up eq.(9) to show that

$$\frac{\Delta\tau_{\text{satellite}}}{\Delta\tau_{\text{ground}}} \simeq 1 + \frac{G_N M_E}{c^2 R_E} - \frac{3 G_N M_E}{2c^2 r}\,, \tag{10}$$

where R_E is Earth's radius and M_E is its mass. Note how the timing discrepancy here can be positive, negative, or zero, depending on r/R_E. How big are these relativistic corrections for Earth's GPS satellites, in SI units?