HW1 Q2: Yang-Mills theory

(a) Prove the Bianchi identity for Yang-Mills gauge fields

$$D_\rho F_{\mu\nu} + D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} = 0 \tag{1}$$

by using lecture material on gauge-covariant derivatives and the adjoint representation. *Hint: start by writing out the LHS with all gauge indices explicit. Organize the terms in powers of g. You will find order g^0, g^1, and g^2 pieces; each term should vanish separately.*

(b) For $SU(N)$ in the fundamental representation only, the product of two generators can be written as

$$T^A T^B = \frac{1}{2N}\delta^{AB} + \frac{1}{2}d^{ABC}T^C + \frac{i}{2}f^{ABC}T^C\,. \tag{2}$$

Here, N is the dimension of the fundamental representation, and d^{ABC} is a group invariant that is symmetric under exchange of any two of its indices. Use eq.(2) to show that

$$d^{ABC} = 2\,\mathrm{Tr}\left(T^A\{T^B, T^C\}\right)\,, \tag{3}$$

and that

$$\mathrm{Tr}\left(T^A T^B T^C\right) = \frac{1}{4}\left(d^{ABC} + if^{ABC}\right)\,, \tag{4}$$

$$\mathrm{Tr}\left(T^A T^B T^C T^D\right) = \frac{1}{4N}\delta^{AB}\delta^{CD} + \frac{1}{8}\left(d^{ABE} + if^{ABE}\right)\left(d^{CDE} + if^{CDE}\right)\,. \tag{5}$$

These formulæ and their generalizations are useful in computing scattering amplitudes for $SU(N)$ gauge fields coupled to charged fundamental matter. For $SU(2)$, $d^{ABC} \equiv 0$. We will make use of this fact when we study chiral anomalies at the end of the course.

(c) What is a **Wilson loop** for a non-Abelian gauge theory? Why is it physically interesting? Why is the exponential involved in its definition path-ordered?
Hint: do not bore me by copying passages out of Peskin & Schroeder. Consult two or three separate QFT textbooks to learn about Wilson loops, and craft an answer in your own words, being careful to maintain proper academic citation hygiene.