PHY484S/1484S GR1 (2018-19) – HW2 – due @ 10am M25Feb2019

Be sure to honestly show your working, to enable awarding of full/partial credit.

2.1: When do FRW universes start contracting again?

(a) `HEL 15.12`: Consider a FRW universe that expands after it goes bang at $t = 0$. Explain why the condition for the $a(t)$ curve to have a turning point is

$$f(a) \equiv \Omega_{\Lambda,0} a^3 + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})a + \Omega_{m,0} = 0\,. \tag{1}$$

For $\Omega_{\Lambda,0} > 0$ like in our cosmos, show by evaluating $f'(a), f''(a)$ that the condition for $f(a)$ to have a single positive root at a_* is $f(a_*) = 0 = f'(a_*)$, and that

$$a_* = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3}\,. \tag{2}$$

Hence show that the values of $(\Omega_{m,0}, \Omega_{\Lambda,0})$, along any dividing line in this plane that separates those models with a turning point from those without, must satisfy

$$4(1 - \Omega_{m,0} - \Omega_{\Lambda,0})^3 + 27\Omega_{m,0}^2 \Omega_{\Lambda,0} = 0\,. \tag{3}$$

(b) `HEL 15.19`: For a radiation-only FRW model with $\Omega_{r,0} \neq 1$, show that

$$a(t) = \left(2H_0 \Omega_{r,0}^{1/2} t\right)^{1/2} \left(1 + \frac{1 - \Omega_{r,0}}{2\Omega_{r,0}^{1/2}} H_0 t\right)^{1/2}\,. \tag{4}$$

Hence, for $\Omega_{r,0} > 1$, show that the $a(t)$ curve has a maximum at

$$a_{\max} = \left(\frac{\Omega_{r,0}}{\Omega_{r,0} - 1}\right)^{1/2}\,, \quad t_{\max} = \frac{1}{H_0} \frac{\Omega_{r,0}^{1/2}}{(\Omega_{r,0} - 1)}\,, \tag{5}$$

and that the age t_0 of such a universe is given by

$$t_0 = \frac{1}{H_0} \frac{1}{\Omega_{r,0}^{1/2} + 1} < \frac{1}{2H_0}\,. \tag{6}$$

PLEASE TURN OVER

2.2: Power-law inflation

(a) `HEL 16.5`: For a potential of the form

$$V(\varphi) = V_0 e^{-\lambda\varphi}\,, \quad \lambda > 0\,, \tag{7}$$

show that the inflation equations can be solved exactly to give

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/\lambda^2}\,, \qquad \varphi = \varphi_0 + \frac{2}{\lambda}\ln\left(\sqrt{\frac{V_0}{2(6-\lambda^2)}}\lambda^2 t\right)\,. \tag{8}$$

Hence show that, provided $\lambda < \sqrt{2}$, the solution corresponds to a period of inflation. Show further that the slow-roll parameters for this model are $\epsilon = \eta/2 = \lambda^2/2$, and so the inflationary epoch never ends. This model is known as power-law inflation.

(b) `HEL 16.7`: In the slow-roll approximation, show that

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2\,. \tag{9}$$

Assuming that $\dot{\varphi}$ varies monotonically with t throughout the period of inflation, show that

$$\dot{\varphi} = -2H'(\varphi)\,, \tag{10}$$

where H is now considered as a function of φ, and hence that we may write the cosmological field equation as

$$[H'(\varphi)]^2 - \frac{3}{2}H^2(\varphi) = -\frac{1}{2}V(\varphi)\,. \tag{11}$$

This is known as the Hamilton-Jacobi formalism for inflation.

(c) `HEL 16.8`: Repeat part (a) using the Hamilton-Jacobi formalism of (b).

PLEASE TURN OVER

2.3: Ultra-relativistic and non-relativistic Bose and Fermi gases

(a) Starting from the expressions (10.2),(10.7–10.10) of lecture notes for n, ρ, P, and assuming that $T \gg m$ and $\mu \simeq 0$, derive the formulae in the first column of the table above eq. (10.11):-

what	ultra-relativistic bosons	ultra-relativistic fermions	non-relativistic
n_A	$1 g_A \frac{\zeta(3)}{\pi^2} T^3$	$\frac{3}{4} g_A \frac{\zeta(3)}{\pi^2} T^3$	$g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} e^{-m_A/T}$
ρ_A	$1 g_A \frac{\pi^2}{30} T^4$	$\frac{7}{8} g_A \frac{\pi^2}{30} T^4$	$m_A n_A$
P_A	$\frac{1}{3}\rho_A$	$\frac{1}{3}\rho_A$	$n_A T \ll \rho_A$

(b) Now derive the formulae in the second column.

(c) Now derive the formulae in the third column.

(d) For $\mu = 0$, the number of particles n and the number of antiparticles $\bar{n}$ of any given species are equal. To allow for a net particle number, we should restore μ. Using the Fermi-Dirac distribution function for $\mu \neq 0$, show that

$$n - \bar{n} = \frac{1}{6\pi^2} g T^3 \left[\pi^2 \left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3 \right] . \tag{12}$$

Note that the RHS here is exact and not a truncation of a series. To solve this part, you will need to use an identity for the PolyLog function.