PHY484S/1484S GR1 (2018-19) – HW1 – due @ 10am M04Feb2019

1.1: Geodesic equation from energy-momentum conservation

The principle of covariant conservation of energy-momentum can be used to derive the geodesic equation for a point particle providing the source of energy-momentum.

(a) Starting from the Einbein action

$$S_{\rm einbein} = \int d\lambda \frac{1}{2}\left[e^{-1}(\lambda)\dot{z}^2(\lambda) + e(\lambda)m^2\right] \,, \tag{1}$$

and the definitions

$$S_{\rm matter} \equiv \int d^D x \, \mathscr{L}_{\rm matter} \,, \quad T_{\alpha\beta} \equiv \frac{2}{\sqrt{-g}}\frac{\delta \mathscr{L}_{\rm matter}}{\delta g^{\alpha\beta}} \,, \tag{2}$$

show that

$$T_{\mu\nu}(x) = \frac{m}{\sqrt{-g(x)}}\int d\lambda \, \dot{z}_\mu(\lambda)\dot{z}_\nu(\lambda)\, \delta^{(D)}(x - z(\lambda)) \,. \tag{3}$$

where in the massless case we really mean 1 instead of m, and $\dot{} \equiv d/d\lambda$.
Hint: you may fix the einbein via $e^{-1}(\lambda) = m$ (massive, 'proper time gauge') or $e^{-1}(\lambda) = 1$ (massless), where λ is the affine parameter for the particle path $z^\mu(\lambda)$.

(b) Show that for an arbitrary rank (2,0) tensor T

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\sigma\mu}T^{\sigma\nu} + \Gamma^\nu{}_{\sigma\mu}T^{\mu\sigma} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,T^{\mu\nu}\right) + \Gamma^\nu{}_{\sigma\mu}T^{\mu\sigma} \,. \tag{4}$$

(c) Using the Einstein equations (including the cosmological constant), the Bianchi identity for the Riemann tensor, and metric compatibility of the Christoffel connection, show that any energy-momentum tensor coupled to gravity must be covariantly conserved,

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{5}$$

Then, by using the chain rule for differentiation

$$\dot{z}^\mu(\lambda)\frac{\partial}{\partial z^\mu(\lambda)}\delta^D(x - z(\lambda)) = \frac{d}{d\lambda}\delta^D(x - z(\lambda)) \,, \tag{6}$$

and recruiting integration by parts, show that covariant conservation of the particle energy-momentum requires

$$\nabla_\mu T^{\mu\nu} = \int d\lambda \left[\frac{d^2 z^\nu(\lambda)}{d\lambda^2} + \Gamma^\nu{}_{\alpha\beta}(z(\lambda))\frac{dz^\alpha(\lambda)}{d\lambda}\frac{dz^\beta(\lambda)}{d\lambda}\right]\delta^D(x - z(\lambda)) = 0 \,. \tag{7}$$

Therefore, the Einstein equations require the geodesic equations for the particle.