Sample questions, combined from 4 past midterms, for study purposes only

University of Toronto Faculty of Arts and Science
Midterm Examinations Fall ****

PHY483H1-F-LEC0101
PHY483H1-F-LEC2001
PHY1483H-F-LEC0101

Relativity Theory I

Weighting: 20% of final grade.
Duration: two hours.
Aids allowed: none.
Formula sheet is provided on page 3. $\longrightarrow$
Total marks: 40.

Instructions

Each part of each question can be attempted independently.
Read a question carefully before answering it.
Allocate your time according to grade weighting.
On conceptual questions, we grade for thoroughness as well as for correctness.
On calculational questions, state your reasoning to enable awarding partial credit.

- Explain the physical significance of the Equivalence Principle.
- In curved spacetime, why is the partial derivative of a tensor not a tensor? Explain how to build a covariant derivative that *is* a tensor.
- What is a geodesic? Explain why geodesics are physically important.
- Explain the physical significance of the geodesic deviation equation.
- How do ocean tides work? Include a sketch, and indicate how Einstein's GR analysis of tidal forces upgrades Newton's.

In full General Relativity, the covariant electromagnetic field strength tensor is defined by

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \,. \tag{1}$$

- Briefly summarize why eq.(1) implies that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{2}$$

- By raising both indices of eq.(1) using the upstairs metric, explain carefully why in curved spacetime

$$F^{\mu\nu} \neq \partial^\mu A^\nu - \partial^\nu A^\mu \,. \tag{3}$$

Consider the tensor defined by

$$T_{\mu\nu} = F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \,. \tag{4}$$

- Compute its trace $T^\mu{}_\mu$, and show that it is zero only in four spacetime dimensions.
- Specialize to $D = 3+1$ flat Minkowski spacetime in Cartesian coordinates. Find the scalar $F^{\rho\sigma}F_{\rho\sigma}$ and then the component T_{00}, in terms of $\vec{E}$ and $\vec{B}$. Do you recognize the form of T_{00}?

Consider the spacetime

$$ds^2 = \left[\cosh^2 r \, dt^2 - dr^2 - \sinh^2 r \, d\phi^2\right] \,. \tag{5}$$

- Show that the nonzero Christoffels are

$$\Gamma^t{}_{tr} (= \Gamma^t{}_{rt}) = \frac{\sinh r}{\cosh r} \,, \qquad \Gamma^r{}_{tt} = \sinh r \cosh r \,, \qquad \Gamma^\phi{}_{\phi r} (= \Gamma^\phi{}_{r\phi}) \,, \qquad \Gamma^r{}_{\phi\phi} \,. \tag{6}$$

  Focus on finding $\Gamma^t{}_{tr}$ and $\Gamma^r{}_{tt}$. The others will not be needed for the rest of this problem.

- Using the above Christoffels, show that the geodesic equations for purely radial motion are

$$\frac{d^2 t}{d\lambda^2} + 2 \frac{dt}{d\lambda} \frac{dr}{d\lambda} \frac{\sinh r}{\cosh r} = 0 \,, \tag{7}$$

$$\frac{d^2 r}{d\lambda^2} + \left(\frac{dt}{d\lambda}\right)^2 \sinh r \cosh r = 0 \,. \tag{8}$$

- Which of these equations has a first integral and why?
- Specialize further to the case of radial null geodesics. Recruit the nullness condition and the first integral above to find $dr/d\lambda$. Integrate it to get $r(\lambda)$, and then use that to find $t(\lambda)$. Your solutions should obey the above geodesic equations.

Formula sheet

Electric and magnetic fields in $D = 3+1$ flat Minkowski spacetime with Cartesian coordinates:-

$$F_{0i} = +\delta_{ij} E^j \,, \; F_{ij} = -\mathfrak{E}_{ijk} B^k \,, \text{ where } \mathfrak{E}_{ijk} = +1(-1) \text{ if } ijk \text{ even (odd) perm. of 123 or 0 otherwise} \tag{9}$$

Covariant permutation symbol:-

$$\mathfrak{E}_{\mu\nu\lambda\sigma} = +1(-1) \text{ if } \mu\nu\lambda\sigma \text{ even (odd) perm. of 0123 or 0 otherwise} \tag{10}$$

Coordinate transformation law for rank (m,n) tensors V:

$$V^{\mu_1{}' \ldots \mu_m{}'}{}_{\nu_1{}' \ldots \nu_n{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\lambda_1}} \cdots \frac{\partial x^{\mu_m{}'}}{\partial x^{\lambda_m}} \frac{\partial x^{\sigma_1}}{\partial x^{\nu_1{}'}} \cdots \frac{\partial x^{\sigma_n}}{\partial x^{\nu_n{}'}} \; V^{\lambda_1 \ldots \lambda_m}{}_{\sigma_1 \ldots \sigma_n} \tag{11}$$

Indices are lowered with $g_{\mu\nu}$ and raised with $g^{\mu\nu}$, where $g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma$ and $g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu$, e.g. for vector V:

$$V_\mu = g_{\mu\nu} V^\nu \tag{12}$$

Downstairs covariant derivative of rank (m,n) tensor V:

$$\begin{aligned} \nabla_\sigma V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = \partial_\sigma V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} &+ \Gamma^{\mu_1}{}_{\sigma\lambda} V^{\lambda \mu_2 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} + \Gamma^{\mu_2}{}_{\sigma\lambda} V^{\mu_1 \lambda \mu_3 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} + \ldots \\ &- \Gamma^{\lambda}{}_{\sigma\nu_1} V^{\mu_1 \ldots \mu_m}{}_{\lambda \nu_2 \ldots \nu_n} - \Gamma^{\lambda}{}_{\sigma\nu_2} V^{\mu_1 \ldots \mu_m}{}_{\nu_1 \lambda \nu_3 \ldots \nu_n} + \ldots \end{aligned} \tag{13}$$

Christoffel symbols, which are symmetric in $\nu \leftrightarrow \sigma$:

$$\Gamma^{\mu}{}_{\nu\sigma} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\nu\sigma} \right) \tag{14}$$

Metric compatibility condition for Christoffel connection:

$$\nabla_\sigma g_{\mu\nu} = 0 \tag{15}$$

Geodesic equation for $x^\mu(\lambda)$, where λ is affine parameter:

$$\frac{D}{d\lambda} \left(\frac{dx^\mu}{d\lambda} \right) = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}{}_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \tag{16}$$

Alternative form of geodesic equation:

$$\frac{d}{d\lambda} \frac{dx_\mu}{d\lambda} = \frac{1}{2} (\partial_\mu g_{\nu\sigma}) \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} \tag{17}$$

Riemann tensor:

$$R^{\rho}{}_{\sigma\mu\nu} = -\partial_\mu \Gamma^{\rho}{}_{\nu\sigma} + \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} - \Gamma^{\rho}{}_{\mu\lambda} \Gamma^{\lambda}{}_{\nu\sigma} + \Gamma^{\rho}{}_{\nu\lambda} \Gamma^{\lambda}{}_{\mu\sigma} \tag{18}$$

Riemann from commutator of covariant derivatives on a rank (k,ℓ) tensor T:

$$\begin{aligned} \left[\nabla_\rho, \nabla_\sigma \right] T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} = &- R^{\mu_1}{}_{\lambda\rho\sigma} T^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} - R^{\mu_2}{}_{\lambda\rho\sigma} T^{\mu_1 \lambda \mu_3 \ldots \mu_k}{}_{\nu_1 \ldots \nu_\ell} + \ldots \\ &+ R^{\lambda}{}_{\nu_1\rho\sigma} T^{\mu_1 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_\ell} + R^{\lambda}{}_{\nu_2\rho\sigma} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \lambda \nu_3 \ldots \nu_\ell} - \ldots \end{aligned} \tag{19}$$

Symmetries of Riemann:

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \,, \qquad R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \,, \qquad R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \,, \qquad R_{[\alpha\beta\gamma\delta]} = 0 \tag{20}$$

Geodesic deviation equation, where S is separation vector and T is tangent vector:

$$\frac{D^2 S^\mu}{d\lambda^2} = (\nabla_T \nabla_T S)^\mu = -R^{\mu}{}_{\nu\alpha\sigma} T^\nu T^\alpha S^\sigma \tag{21}$$

Newtonian limit, where $x^0 = ct$:-

$$\left| \frac{\vec{v}}{c} \right| \ll 1 \,, \; |\partial_0| \ll |\partial_i| \,, \; \left| \frac{\Phi}{c^2} \right| \ll 1 \,, \; ds^2 = \left(1 + \frac{2\Phi}{c^2} \right) (dx^0)^2 - \left(1 - \frac{2\Phi}{c^2} \right) |d\vec{x}|^2 \,, \; g_{\mu\nu} \simeq \eta_{\mu\nu} + \epsilon h_{\mu\nu} \,, \; g^{\mu\nu} \simeq \eta^{\mu\nu} - \epsilon h^{\mu\nu} \tag{22}$$