PHY483F/1483F final exam info – Dec.2017 – Prof. A.W. Peet

Our final exam will feature four calculation-oriented questions, centred on the following topics:–

1. geodesic deviation;
2. classic experimental tests of GR;
3. basic aspects of black holes;
4. basic aspects of gravitational waves.

Since the exam is closed book, you will be given all the equations you need, either in the question or in the formula sheet on the last page of the exam paper. *(See sample formula sheet on page 3.)*⟶

►► Below are six sample problems for you to practice solving, to help you study. ◄◄
Some of them are pertinent old midterm/final exam questions (suitably tidied up); others are new. I will not be providing solutions, but you are welcome to ask about these problems in office hours.

A: In flat Minkowski spacetime, the Lagrangian for electromagnetism coupled to a current J^μ is $\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}F_{\lambda\sigma}$. Use the Euler-Lagrange equations for the gauge potential A_σ to show that its equation of motion is $\partial_\mu F^{\mu\nu} = J^\nu$.

B: Consider the 2D sphere line element $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. Calculate Riemann. Write down the two geodesic deviation equations. Examine the case where the tangent vector points down a line of longitude, and show how geodesics separated azimuthally get "pulled together" by the sphere's positive curvature. *(Don't bother with finding an orthonormal frame like in HW4. Just use the ordinary geodesic deviation equations in the coordinate basis listed in the formula sheet on page 3.)*

C: Suppose that we live in a 3D spacetime with coordinates $\{t, r, \phi\}$ and line element given by $ds^2 = f(r)dt^2 - [f(r)]^{-1}dr^2 - r^2 d\phi^2$, where $f(r) = 1 \pm r^2/\ell^2$ and $\ell = \text{const}$. Show that two symmetries of the metric imply conservation laws: $E = f(r)\,\dot{t} = \text{const}$ and $L = r^2\dot{\phi} = \text{const}$, where $\dot{} \equiv d/d\lambda$. Using E, L, and the mass shell constraint for a particle moving in this curved spacetime, find an expression for $\dot{r}^2$. Recruit effective potential language to figure out whether circular orbits are possible for (i) massless and (ii) massive particles. Does the answer depend on the sign in $f(r)$?

D: Describe the primary anatomical features of the stationary Kerr black hole spacetime

$$ds^2 = \left(1 - \frac{2\mu r}{\sigma^2}\right) dt^2 + \frac{4\mu a r \sin^2\theta}{\sigma^2}\, dt\, d\phi - \frac{\sigma^2}{\Delta} dr^2 - \sigma^2 d\theta^2 - \left(r^2 + a^2 + \frac{2\mu a^2\, r \sin^2\theta}{\sigma^2}\right) \sin^2\theta\, d\phi^2\,,$$

where $\sigma^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2\mu r + a^2$. Don't forget to mention where the horizon(s) and stationary limit surface (ergosphere) are located. Include a sketch.

E: Check that you understand the more elementary material from my gravitational waves lectures by doing the straightforward exercises I recommended in class. *(Don't bother grappling with the fine details of the retarded time story or of deriving the energy loss formula for gravitational radiation: these are too advanced/lengthy to test in a closed book exam.)*

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F: As you know, the Einstein equations imply that any energy-momentum tensor $T^{\mu\nu}$ must be covariantly conserved: $\nabla_\mu T^{\mu\nu} = 0$. Using the metric determinant formula *(see formula sheet)*, show that for any rank (2,0) tensor T

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\sigma\mu} T^{\sigma\nu} + \Gamma^\nu{}_{\sigma\mu} T^{\mu\sigma} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, T^{\mu\nu} \right) + \Gamma^\nu{}_{\sigma\mu} T^{\mu\sigma} \,.$$

For a relativistic point particle of mass $m > 0$ in curved spacetime,

$$T^{\mu\nu}_{\text{particle}}(x) = m \int d\lambda \frac{\delta^D(x - z(\lambda))}{\sqrt{-g(x)}} \frac{dz^\mu(\lambda)}{d\lambda} \frac{dz^\nu(\lambda)}{d\lambda} \,.$$

By using the chain rule for differentiation

$$\frac{d}{d\lambda} \delta^D(x - z(\lambda)) = \dot{z}^\mu(\lambda) \frac{\partial}{\partial z^\mu(\lambda)} \delta^D(x - z(\lambda)) \,,$$

and recruiting integration by parts, show that covariant conservation of energy-momentum for the massive particle requires

$$\nabla_\mu T^{\mu\nu}_{\text{particle}} = \int d\lambda \left[\frac{d^2 z^\nu(\lambda)}{d\lambda^2} + \Gamma^\nu{}_{\alpha\beta}(z(\lambda)) \frac{dz^\alpha(\lambda)}{d\lambda} \frac{dz^\beta(\lambda)}{d\lambda} \right] \delta^D(x - z(\lambda)) = 0 \,.$$

Conclude that the Einstein equations require the geodesic equations for the particle, with affine parameter λ. *(Note: there is no analogue of this story for electromagnetism, a linear theory.)*

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Formula sheet

particle mass shell constraint : $$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \epsilon\,, \text{where } \epsilon = 1 \text{ (timelike) or } 0 \text{ (null)}\,. \tag{1}$$

momentum/velocity conventions : $$p^\mu = mu^\mu = m\frac{dx^\mu}{d\lambda} \text{ (timelike)}\,;\; p^\mu = \frac{dx^\mu}{d\lambda} \text{ (null)}\,. \tag{2}$$

photon energy : $$E = p_\mu u^\mu\,, \text{ for photon with null } p^\mu \text{ and observer with timelike } u^\mu\,. \tag{3}$$

tensor transformation law : $$W^{\mu_1{}'\ldots\mu_m{}'}{}_{\nu_1{}'\ldots\nu_n{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\rho_1}}\ldots\frac{\partial x^{\mu_m{}'}}{\partial x^{\rho_m}}\frac{\partial x^{\sigma_1}}{\partial x^{\nu_1{}'}}\ldots\frac{\partial x^{\sigma_n}}{\partial x^{\nu_n{}'}}W^{\rho_1\ldots\rho_m}{}_{\sigma_1\ldots\sigma_n}\,. \tag{4}$$

Christoffel connection : $$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)\,; \qquad \Gamma^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\nu\mu}\,. \tag{5}$$

metric compatibility : $$\nabla_\sigma g_{\mu\nu} = 0\,. \tag{6}$$

covariant derivative of tensor : $$\nabla_\sigma W^{\mu_1\cdots\mu_m}{}_{\nu_1\ldots\nu_n} = \partial_\sigma W^{\mu_1\cdots\mu_m}{}_{\nu_1\ldots\nu_n} \tag{7}$$

$$+\,\Gamma^{\mu_1}_{\sigma\rho}W^{\rho\mu_2\cdots\mu_m}{}_{\nu_1\ldots\nu_n} + \Gamma^{\mu_2}_{\sigma\rho}W^{\mu_1\rho\mu_3\cdots\mu_m}{}_{\nu_1\ldots\nu_n} + \ldots \tag{8}$$

$$-\,\Gamma^\rho{}_{\sigma\nu_1}W^{\mu_1\cdots\mu_m}{}_{\rho\nu_2\ldots\nu_n} - \Gamma^\rho{}_{\sigma\nu_2}W^{\mu_1\cdots\mu_m}{}_{\nu_1\rho\nu_3\ldots\nu_n} - \ldots\,. \tag{9}$$

geodesic equations : $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0\,, \quad \text{where } \lambda \text{ is affine parameter}\,. \tag{10}$$

alternative form of geodesic equations : $$\frac{dp_\mu}{d\lambda} = \frac{1}{2}\left(\partial_\mu g_{\nu\sigma}\right)p^\nu p^\sigma\,. \tag{11}$$

Killing vector equations : $$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad \Rightarrow \quad p^\mu\nabla_\mu(K_\nu p^\nu) = 0\,. \tag{12}$$

Riemann tensor : $$R^\rho{}_{\sigma\mu\nu} = -\partial_\mu\Gamma^\rho{}_{\nu\sigma} + \partial_\nu\Gamma^\rho{}_{\mu\sigma} - \Gamma^\rho{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\sigma} + \Gamma^\rho{}_{\nu\alpha}\Gamma^\alpha{}_{\mu\sigma}\,. \tag{13}$$

Riemann symmetries : $$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}\,, \qquad R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}\,, \qquad R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}\,. \tag{14}$$

Ricci tensor and Ricci scalar : $$R_{\mu\nu} = -R^\alpha{}_{\mu\alpha\nu}\,, \quad R = g^{\mu\nu}R_{\mu\nu}\,. \tag{15}$$

Bianchi identity for Einstein tensor : $$\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0\,. \tag{16}$$

geodesic deviation : $$\frac{D^2S^\mu}{D\lambda^2} = -R^\mu{}_{\nu\alpha\sigma}T^\nu T^\alpha S^\sigma\,, \quad \text{where } S^\mu \text{ is separation}\,,\ T^\mu \text{ is tangent}\,. \tag{17}$$

Einstein equations : $$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G_N}{c^4}T_{\mu\nu}\,, \quad \text{with} \quad \nabla_\mu T^{\mu\nu} = 0\,. \tag{18}$$

stationary black hole : $$\text{horizon(s)} : g^{rr} = 0\,, \quad \text{stationary limit surface} : g_{tt} = 0\,. \tag{19}$$

perfect fluid : $$T^{\mu\nu}_{\text{p.f.}} = \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - pg^{\mu\nu}\,. \tag{20}$$

gravitational waves : $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \quad g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}\,, \quad \text{where} \quad h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}\,; \tag{21}$$

$$\Box^2\bar{h}^{\mu\nu} = -\frac{16\pi G_N}{c^4}T^{\mu\nu}\,, \text{ where } \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \text{ and } \partial_\mu\bar{h}^{\mu\nu} = 0\,. \tag{22}$$

metric determinant : $$\Gamma^\mu{}_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\right)\,, \quad \text{where} \quad (-g) = -\det(g_{\alpha\beta})\,. \tag{23}$$