# PHY483F (2017-18) – HW4 – due 11am Thu.30.Nov

For this homework, we will try something different suggested by the TA: digesting a classic influential paper from the research literature. He chose one by a leading relativist, Sir Roger Penrose, from 1969:-

http://ap.io/483f/files/penrose1969.pdf

N.B.: you are encouraged to discuss Penrose's paper and the questions below with other students, but you must write up independent solutions for each part by yourself.

## 4.1 Summarizing

Write a summary of the main physics points in Penrose's paper, in your own words. Length: 1200-1500 words.

(How do you determine the word count of an equation you want to include? Select a good-sized chunk of representative non-equation text written by you, and compute[1] your average number of words per line $\overline{N}$. Then measure what fraction $f$ of a line your equation takes up on the page. The word count of your equation is $f\,\overline{N}$.)

## 4.2 Figuring out jargon

For this question, you will need to do a little extra research using other sources. Be sure to cite them properly.

(a) What is a **trapped surface**?

(b) What is a **closed timelike curve** (CTC), and why is it physically pathological?

(c) What is the **cosmic censorship hypothesis**?

## 4.3 Calculating when gravity is attractive

The key tool needed to figure out when gravity is attractive is the Raychaudhuri equations describing the expansion of a congruence of geodesics, which I derived in the Appendix of lecture notes. For our purposes, we can assume hypersurface orthogonality so that the $\omega_{\mu\nu}\omega^{\mu\nu}$ term vanishes. Then for the case of timelike geodesics, the expansion $\theta$ of the congruence obeys

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}\,u^\mu u^\nu\,, \tag{1}$$

where $u^\mu$ is the timelike 4-velocity vector, and the shear $\sigma_{\mu\nu}$ obeys $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$. For the case of null geodesics, the expansion $\hat{\theta}$ of the congruence obeys

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2}\hat{\theta}^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + R_{\mu\nu}\,k^\mu k^\nu\,, \tag{2}$$

where $k^\mu$ is a null vector and the shear $\hat{\sigma}_{\mu\nu}$ obeys $\hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} \geq 0$.

(a) Recall the handy alternative form of the Einstein equations that you derived in a previous homework,

$$R_{\mu\nu} = -8\pi G_N\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \equiv -8\pi G_N\Delta_{\mu\nu}\,. \tag{3}$$

Use the perfect fluid form of the energy-momentum tensor to find an expression for $\Delta_{\mu\nu}$ in terms of $\rho$, $p$, $u^\mu$.

(b) By contracting $\Delta_{\mu\nu}$ with $u^\mu u^\nu$ and $k^\mu k^\nu$ respectively, show that for the expansion of timelike and null geodesic congruences to be non-positive, we need the energy density and pressure to obey

$$\rho + 3\frac{p}{c^2} \geq 0 \quad \text{and} \quad \rho + \frac{p}{c^2} \geq 0\,. \tag{4}$$

This is known as the **Strong Energy Condition**. It is not obeyed by all forms of matter/energy in our universe. In particular, the first condition is disobeyed by the cosmological constant.

[1] Handy word counters for electronic text can be found online, e.g. at http://wordcounter.net/

(Please turn over)

### 4.4 Calculating tidal forces for Schwarzschild

When I defined tensors in curved spacetime way back near the start of my introduction to GR, I briefly mentioned that there is a coordinate-independent way of working with tensors which mathematicians tend to favour. When we use the *coordinate* basis tensors $\partial_\mu$ and $dx^\mu$ for writing out tensor components, we recover our usual physicist index notation. For calculating tidal forces on objects falling into Schwarzschild black holes, it turns out to be easier to work in an *orthonormal* basis instead of the coordinate basis (where the coordinate singularity at $r = 2\mu$ clouds the physics interpretation). In an orthonormal basis, the geodesic deviation equation simplifies to

$$\frac{d^2}{d\lambda^2} S^{\hat{\alpha}} = +R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} T^{\hat{\beta}} T^{\hat{\gamma}} S^{\hat{\delta}} \,. \tag{5}$$

For any contravariant vector $V$, the hatted components are obtained from the unhatted ones via

$$V^{\hat{\alpha}} = (e^{\hat{\alpha}})_\mu V^\mu \,, \tag{6}$$

where the $(e^{\hat{\alpha}})_\mu$ are defined by

$$g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} (e^{\hat{\alpha}})_\mu (e^{\hat{\beta}})_\nu \,. \tag{7}$$

For the rank (1,3) Riemann tensor,

$$R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} = R^\mu{}_{\nu\sigma\rho} (e^{\hat{\alpha}})_\mu (e_{\hat{\beta}})^\nu (e_{\hat{\gamma}})^\sigma (e_{\hat{\delta}})^\rho \,, \tag{8}$$

where the $(e_{\hat{\alpha}})^\mu$ are the inverses of $(e^{\hat{\alpha}})_\mu$:

$$(e_{\hat{\alpha}})^\nu (e^{\hat{\alpha}})_\mu = \delta^\nu_\mu \,, \qquad (e^{\hat{\alpha}})_\mu (e_{\hat{\beta}})^\mu = \delta^{\hat{\alpha}}_{\hat{\beta}} \,. \tag{9}$$

For a massive particle released at rest from radius $r$ in the Schwarzschild geometry in coordinates $\{t, r, \theta, \phi\}$, we can take the $(e^{\hat{\alpha}})_\mu$ and $(e_{\hat{\alpha}})^\mu$ to be

$$\begin{aligned}
(\hat{e}^{\hat{0}})_\mu &= \sqrt{1 - 2\mu/r}\, \delta^{\hat{0}}_\mu \,, & (\hat{e}_{\hat{0}})^\mu &= \frac{1}{\sqrt{1 - 2\mu/r}} \delta^\mu_{\hat{0}} \,, \\
(\hat{e}^{\hat{1}})_\mu &= \frac{1}{\sqrt{1 - 2\mu/r}} \delta^{\hat{1}}_\mu \,, & (\hat{e}_{\hat{1}})^\mu &= \sqrt{1 - 2\mu/r}\, \delta^\mu_{\hat{1}} \,, \\
(\hat{e}^{\hat{2}})_\mu &= r\, \delta^{\hat{2}}_\mu \,, & (\hat{e}_{\hat{2}})^\mu &= \frac{1}{r} \delta^\mu_{\hat{2}} \,, \\
(\hat{e}^{\hat{3}})_\mu &= r \sin\theta\, \delta^{\hat{3}}_\mu \,, & (\hat{e}_{\hat{3}})^\mu &= \frac{1}{r \sin\theta} \delta^\mu_{\hat{3}} \,.
\end{aligned} \tag{10}$$

which describes the inertial instantaneous rest frame (IIRF) of the particle.

(a) Check that eq.(7) is satisfied.

(b) Compute the components of $R^\mu{}_{\nu\sigma\rho}$ in $\{t, r, \theta, \phi\}$ coordinates using Maxima. Then use eq.(8) to find the hatted time-space-time-space components $R^{\hat{1}}{}_{\hat{0}\hat{0}\hat{1}}$, $R^{\hat{2}}{}_{\hat{0}\hat{0}\hat{2}}$, and $R^{\hat{3}}{}_{\hat{0}\hat{0}\hat{3}}$ only. You should find that

$$R^{\hat{1}}{}_{\hat{0}\hat{0}\hat{1}} = +2\frac{\mu}{r^3} \,, \quad R^{\hat{2}}{}_{\hat{0}\hat{0}\hat{2}} = -1\frac{\mu}{r^3} = R^{\hat{3}}{}_{\hat{0}\hat{0}\hat{3}} \,. \tag{11}$$

(c) Assume that two slightly separated particles fall into the black hole purely radially. Find the hatted components of the tangent vector $T^{\hat{\alpha}}$, recalling that we drop the particles in from rest at radius $r$. Then, assuming that the separation vector $S^{\hat{\alpha}}$ is purely spatial, show that by the geodesic deviation equation (5)

$$\frac{d^2}{d\lambda^2} S^{\hat{1}} = +2\frac{\mu}{r^3} S^{\hat{1}} \,, \quad \frac{d^2}{d\lambda^2} S^{\hat{2}} = -1\frac{\mu}{r^3} S^{\hat{2}} \,, \quad \frac{d^2}{d\lambda^2} S^{\hat{3}} = -1\frac{\mu}{r^3} S^{\hat{3}} \,. \tag{12}$$

Physically, this corresponds to stretching along the radial direction and compression (pressure) along angular directions. This is why people sketch infalling astronauts as becoming taller and thinner the closer they get to the singularity, the location at which tidal forces blow up.

(d) Argue from your answers above that tidal forces from smaller black holes are more violent at the horizon than those from large ones. Assuming that the human limit on stretching/compression of body tissues is an acceleration gradient of $\sim 400$ m.s$^{-2}$/m, how small must a Schwarzschild black hole be to ensure that tidal forces kill a freely infalling astronaut even before they cross the event horizon? You should find $M \lesssim 10^5 M_\odot$.