PHY483F/1483F (2017-18) – HW3 – due 11am R16Nov

Note: assume that $D = 3 + 1$ throughout this assignment.

3.1: Dropping a buoy into a Schwarzschild black hole

Suppose that we are on the starship Enterprise and position ourselves at rest at a radius $r_0 > r_S$ away from a Schwarzschild black hole. Then suppose that we drop a buoy directly towards the centre of the black hole.

(a) Derive a formula for the amount of affine (proper) time it takes for the buoy to freefall into the singularity. Make use the first integrals of the geodesic equations for Schwarzschild that we derived in lectures via Killing vectors, and the tangent vector norm condition.

(b) If we had a solar-mass black hole and dropped the buoy in from $r_0 = 10\, r_S$, how long would it take in seconds?

3.2: Energy-momentum for perfect fluids

Consider (HEL8.2) the energy-momentum tensor of a perfect fluid. It encodes information about energy density ρ, pressure p, and fluid 4-velocity u^μ,

$$T^{\mu\nu}_{\rm pf} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu} \,. \tag{1}$$

(a) Show that for any fluid

$$u_\nu \nabla_\mu u^\nu = 0 \,. \tag{2}$$

(b) Using conservation of energy-momentum as required by the Einstein equations,

$$\nabla_\mu T^{\mu\nu} = 0 \,, \tag{3}$$

show that a perfect fluid must obey

$$\begin{aligned} \nabla_\mu(\rho u^\mu) + \frac{p}{c^2}\nabla_\mu u^\mu = 0 \quad \text{and} \\ \left(\rho + \frac{p}{c^2}\right) u^\mu \nabla_\mu u^\nu = \left(g^{\mu\nu} - \frac{u^\mu u^\nu}{c^2}\right)\nabla_\mu p \,. \end{aligned} \tag{4}$$

(c) Give a physical interpretation of these equations (**??**).

(d) Obtain the equation of motion for the worldline $x^\mu(\lambda)$ of a particle in a perfect fluid with pressure, and use it to show that the particle is 'pushed off' geodesics by the pressure gradient.

Hint: if you get stuck in deriving the *pair* of equations (**??**), don't forget that you can always contract a tensor equation you derived with another handy tensor you might have lying around.

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3.3: Cosmological black holes

(a) Show that for a general spacetime, the Einstein equations can be rearranged to

$$R_{\mu\nu} = -8\pi G_N \left[T_{\mu\nu} - \frac{1}{(D-2)} T g_{\mu\nu}\right] + \frac{2}{(D-2)} g_{\mu\nu}\Lambda \,. \tag{5}$$

Hint: start by tracing the Einstein equations to find R.

(b) Suppose (HEL9.27) that the cosmological constant Λ is *nonzero*. By recruiting the same logic that we used in class to derive Birkhoff's theorem without Λ, show that the line element outside a static spherically symmetric matter distribution is

$$ds^2 = \left(1 - \frac{2\mu}{r} - \frac{\Lambda r^2}{3}\right)(cdt)^2 - \left(1 - \frac{2\mu}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 d\Omega_2^2 \,.$$

Feel free to use relevant equations derived in lecture notes without rederiving them yourself, but remember to carefully cite where you got them from.

(c) Show that in the weak-field Newtonian limit, a spherically symmetric mass M produces a gravitational field strength $\vec{g}$ given by

$$\vec{g} = \left(-\frac{G_N M}{r^2} + \frac{c^2 \Lambda r}{3}\right)\hat{r} \,.$$

How large is the cosmological constant correction term on Earth's surface?

(d) Argue from this that the shapes of massive particle orbits in the above geometry differ from those in the Schwarzschild geometry, but that the shapes of photon orbits do not.
Use full GR to do this analysis, not the Newtonian limit.