PHY483F/1483F GR1 (2017-18) – HW1 – due @11am Thu.05.Oct

About my grading philosophy. I never encourage my students to think that they start out entitled to a grade of 100% and lose marks for making mistakes. Penalizing mistakes is not my educational style. Indeed, I frame things in exactly the opposite way: students start out with 0% and earn positive marks for getting physics logic right. Awarding partial credit fairly is one of my central grading principles, so you should **show your working clearly**.

As mentioned in class, I encourage you to discuss *generalities* of the homework problems with other students taking this course for credit, if you like. However, these homeworks are graded individually, so the TA and I need to see specific evidence of individual understanding of each problem in each homework (regulations require us to check). All homeworks must be accompanied by a signed Academic Integrity Declaration or they will not be graded.

1.1: Solving the twin paradox for constant relativistic acceleration

Pretend that an astronaut twin and a homebody twin are alone in the universe and that $G_N = 0$, so that the fabric of spacetime is flat. Model the astronaut's trip as having four parts involving constant relativistic acceleration (CRA):-

1. a CRA rocket burn with $+g$ producing maximum speed $+v_*$, followed by
2. a CRA rocket burn with $-g$ to reach the turnaround point;
3. a CRA rocket burn with $-g$ producing maximum speed $-v_*$, followed by
4. a CRA rocket burn with $+g$ to get back home.

(a) By recruiting ideas from lecture notes on CRA, derive a formula for how much younger the astronaut twin is when they return home than their homebody twin, as a function of g and v_* (and c). Does your expression have the correct non-relativistic limit?

(b) Assuming experimentally optimistic values for g and v_*, what order of magnitude of anti-ageing could present-day rocket entrepreneurs offer wealthy celebrities willing to spend one week in a spaceship? For this part, you will need to derive the elapsed time of the astronaut trip in the homebody frame in terms of g and v_*, and restore factors of c using dimensional analysis.

1.2: Electric/magnetic fields, Maxwell's equations, and Lorentz transformations

In lecture notes, we showed how the electromagnetic field strength tensor $F_{\mu\nu}$ contains the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$,

$$F_{0i} = +\delta_{ij} E^j \,, \qquad F_{ij} = -\mathfrak{E}_{ijk} B^k \,. \tag{1}$$

(a) Starting from the above and the covariant Maxwell equation with a source,

$$\partial_\mu F^{\mu\nu} = J^\nu \,, \tag{2}$$

derive the two 3-vector Maxwell equations for $\vec{E}$ and $\vec{B}$ involving the charge ρ and the current $\vec{j}$.

(b) Check that the Bianchi identity

$$\mathfrak{E}^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0 \tag{3}$$

produces the other two (sourceless) Maxwell's equations in 3-vector form.

(c) By applying the tensorial transformation law for $F_{\mu\nu}$ under boosts, derive the Lorentz transformation rules for $\vec{E}$ and $\vec{B}$ as 3-vector equations. To reduce your algebra burden along the way, separate the electric and magnetic fields into components parallel and perpendicular to the boost direction: $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$ and $\vec{B} = \vec{B}_\parallel + \vec{B}_\perp$. You should find that

$$\begin{aligned} \vec{E}'_\parallel &= \vec{E}_\parallel \,, \\ \vec{B}'_\parallel &= \vec{B}_\parallel \,, \\ \vec{E}'_\perp &= \gamma\left(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp\right) \,, \\ \vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \vec{v}\times\vec{E}_\perp\right) \,, \end{aligned} \tag{4}$$

where $\gamma = 1/\sqrt{1-v^2}$.

1.3: Flat spacetime in curvilinear coordinates, the Christoffel connection, and Maxima

Consider Minkowski spacetime in spherical polar coordinates,

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,. \tag{5}$$

Even though the metric is flat, there is a nontrivial Christoffel connection, because the coordinates are curvilinear.

(a) Compute the Christoffel symbols in spherical polar coordinates, by hand, using

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \,. \tag{6}$$

(b) Do the same calculation using Maxima. Instructions:-

- If you have access to a computer on which you can install software, download Maxima and install it (preferred). If you do not, you can use Maxima Online.
- Use Maxima, *not* Maple, Mathematica, Matlab, or any other software than Maxima. We are standardizing on Maxima for HW1 and future homeworks in order to make the playing field level for all students.
- You must hand in your Maxima file, or an equivalent full record of what you did in Maxima Online, with your homework assignment. Be sure to document what the commands you used actually do, as well as showing the output results.
- If you use someone else's Maxima file as a template for your calculations, you must acknowledge that with a proper citation. Academic citation hygiene requires it.

(Hint: here are some sample Maxima files I wrote for cylindrical polars: interactive version, command line version.)

(c) Using the Christoffel symbols from (a), find the covariant Laplacian of a scalar field $\Psi(t, r, \theta, \phi)$ in spherical polar coordinates, namely $\nabla^\mu \nabla_\mu \Psi$. You should find the resulting expression familiar from multivariable calculus class. (Hint: on a scalar field Ψ, and only on a scalar field, $\nabla_\mu \Psi = \partial_\mu \Psi$.)
