PHY483F/1483F (2018-19) – HW4 – due 11am R29Nov

4.1 How wide is the sun? *[3 marks]*

All massive objects look larger than they really are, because gravity bends light. Consider a light ray grazing the surface of a massive sphere of coordinate radius $r > 3G_N M/c^2$. Show that the light ray will arrive at infinity with impact parameter

$$b = r\left(\frac{r}{r - 2G_N M/c^2}\right)^{1/2} . \tag{1}$$

Hence show that the apparent diameter of the Sun, which has a mass of $M_\odot \sim 2 \times 10^{30}$ kg and a radius of $r_\odot \sim 7 \times 10^5$ km, exceeds the coordinate diameter by nearly 3 km.

4.2 How fast do massive particles move in the Schwarzschild ISCO? *[6 marks]*

Show that a massive particle moving in the innermost stable circular orbit in the Schwarzschild geometry has speed $c/2$ as measured by a stationary observer at this radius. Using this, calculate the period of the orbit as measured by this local observer. What is the period of the orbit as measured by a stationary observer at infinity?

4.3 When is gravity attractive? *[6 marks]*

The key tool needed to figure out when gravity is attractive is the Raychaudhuri equations describing the expansion of a congruence (group) of geodesics. Under certain technical assumptions which do not concern us here, the expansion θ of a congruence of timelike geodesics obeys

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + R_{\mu\nu}\, u^\mu u^\nu , \tag{2}$$

where u^μ is the timelike 4-velocity vector and the shear tensor $\sigma_{\mu\nu}$ obeys $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$. Under similar technical assumptions, the expansion $\hat{\theta}$ of a congruence of null geodesics obeys

$$\frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2}\hat{\theta}^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + R_{\mu\nu}\, k^\mu k^\nu , \tag{3}$$

where k^μ is a null vector and $\hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} \geq 0$.

(a) *[2 marks]* Recall the handy alternative form of the Einstein equations that you derived in HW3[1],

$$R_{\mu\nu} = -8\pi G_N\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \equiv -8\pi G_N \Delta_{\mu\nu} . \tag{4}$$

Assuming the energy-momentum tensor takes perfect fluid form, find an expression for $\Delta_{\mu\nu}$ in terms of the energy density ρ, the pressure p, and the four-velocity field u^μ.

(b) *[4 marks]* Show that for the expansion of timelike and null geodesic congruences to be non-positive, the energy density and pressure must obey

$$\rho + 3\frac{p}{c^2} \geq 0 \quad \text{and} \quad \rho + \frac{p}{c^2} \geq 0 . \tag{5}$$

This is known as the **Strong Energy Condition**. The SEC is not obeyed by all forms of matter/energy in our universe. Is it obeyed or disobeyed by the cosmological constant?

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[1]Here, $T_{\mu\nu}$ counts the energy-momentum from both matter and the cosmological constant, if any.

4.4 When do astronauts falling into Schwarzschild black holes die? *[10 marks]*

To understand tidal forces on an object falling into a Schwarzschild black hole, it is smartest to work in an *orthonormal* basis instead of the usual coordinate basis (where the coordinate singularity at the horizon clouds the physical interpretation). In an orthonormal basis, the geodesic deviation equation simplifies to

$$\frac{d^2}{d\lambda^2} S^{\hat{\alpha}} = +R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} T^{\hat{\beta}} T^{\hat{\gamma}} S^{\hat{\delta}} \,. \tag{6}$$

For any contravariant vector V or covariant vector ω, the hatted components in an orthonormal basis are obtained from the unhatted ones we are used to via

$$V^{\hat{\alpha}} = (e^{\hat{\alpha}})_\mu V^\mu \,, \quad \omega_{\hat{\alpha}} = (e_{\hat{\alpha}})^\mu \omega_\mu \,, \tag{7}$$

where the $(e^{\hat{\alpha}})_\mu$ and $(e_{\hat{\alpha}})^\mu$ are defined by

$$g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} (e^{\hat{\alpha}})_\mu (e^{\hat{\beta}})_\nu \,, \quad g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}} (e_{\hat{\alpha}})^\mu (e_{\hat{\beta}})^\nu \,. \tag{8}$$

The $(e_{\hat{\alpha}})^\mu$ are the inverses of $(e^{\hat{\alpha}})_\mu$:

$$(e_{\hat{\alpha}})^\nu (e^{\hat{\alpha}})_\mu = \delta^\nu_\mu \,, \qquad (e^{\hat{\alpha}})_\mu (e_{\hat{\beta}})^\mu = \delta^{\hat{\alpha}}_{\hat{\beta}} \,. \tag{9}$$

For the rank (1,3) Riemann tensor, the hatted components are obtained from the unhatted ones via a straightforward generalization of eq.s (7),

$$R^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}\hat{\delta}} = R^\mu{}_{\nu\sigma\rho} (e^{\hat{\alpha}})_\mu (e_{\hat{\beta}})^\nu (e_{\hat{\gamma}})^\sigma (e_{\hat{\delta}})^\rho \,. \tag{10}$$

For a massive particle released at rest from radius r in the Schwarzschild geometry for coordinates $\{t, r, \theta, \phi\}$, we can take the $(e^{\hat{\alpha}})_\mu$ to be

$$(e^{\hat{0}})_\mu = \sqrt{1 - 2\mu/r}\, \delta^{\hat{0}}_\mu \,, \quad (e^{\hat{1}})_\mu = \frac{1}{\sqrt{1 - 2\mu/r}}\, \delta^{\hat{1}}_\mu \,, \quad (e^{\hat{2}})_\mu = r\, \delta^{\hat{2}}_\mu \,, \quad (e^{\hat{3}})_\mu = r \sin\theta\, \delta^{\hat{3}}_\mu \,, \tag{11}$$

which describes the inertial instantaneous rest frame (IIRF) of the particle.

(a) *[3 marks]* Show that the $(e^{\hat{\alpha}})_\mu$ in eq. (11) satisfy the first set of eq.s (8). Then, using eq.s (9) and the second set of eq.s (8), show that the $(e_{\hat{\beta}})^\mu$ are given by

$$(e_{\hat{0}})^\mu = \frac{1}{\sqrt{1 - 2\mu/r}}\, \delta^\mu_{\hat{0}} \,, \quad (e_{\hat{1}})^\mu = \sqrt{1 - 2\mu/r}\, \delta^\mu_{\hat{1}} \,, \quad (e_{\hat{2}})^\mu = \frac{1}{r}\, \delta^\mu_{\hat{2}} \,, \quad (e_{\hat{3}})^\mu = \frac{1}{r \sin\theta}\, \delta^\mu_{\hat{3}} \,. \tag{12}$$

(b) *[3 marks]* Compute the components of $R^\mu{}_{\nu\sigma\rho}$ in $\{t, r, \theta, \phi\}$ coordinates using Maxima. Then use eq. (10) to find the hatted components of Riemann. You should find that

$$R^{\hat{1}}{}_{\hat{0}\hat{0}\hat{1}} = +2\frac{\mu}{r^3} \,, \quad R^{\hat{2}}{}_{\hat{0}\hat{0}\hat{2}} = -1\frac{\mu}{r^3} = R^{\hat{3}}{}_{\hat{0}\hat{0}\hat{3}} \,, \tag{13}$$

and three other equations.

(c) *[2 marks]* Assume that two slightly separated particles fall into the black hole purely radially. Find the hatted components of the tangent vector $T^{\hat{\alpha}}$, recalling that we drop the particles in from rest at radius r. Then, assuming that the separation vector $S^{\hat{\alpha}}$ is purely spatial, show that by the geodesic deviation equation (6)

$$\frac{d^2}{d\lambda^2} S^{\hat{1}} = +2\frac{\mu}{r^3} S^{\hat{1}} \,, \quad \frac{d^2}{d\lambda^2} S^{\hat{2}} = -1\frac{\mu}{r^3} S^{\hat{2}} \,, \quad \frac{d^2}{d\lambda^2} S^{\hat{3}} = -1\frac{\mu}{r^3} S^{\hat{3}} \,. \tag{14}$$

Physically, this corresponds to stretching along the radial direction and compression (pressure) along angular directions. This is why people sketch infalling astronauts as becoming taller and thinner the closer they get to the singularity at $r = 0$, the location at which tidal forces blow up – they end up getting spaghettified!

(d) *[2 marks]* Argue from your answers above that tidal forces from smaller black holes are more violent at the horizon than those from large ones. Assuming that the human limit on stretching/compression of body tissues is an acceleration gradient of $\sim 400\, ms^{-2}/m$, how small must a Schwarzschild black hole be to ensure that tidal forces kill a freely infalling astronaut even before they cross the event horizon? You should find $M \lesssim 10^5 M_\odot$.