PHY483F/1483F GR1 (2018-19) HW1 due @11am Thu.04.Oct *[total: 25 marks]*

A note about my grading philosophy. I never encourage my students to think that they start out entitled to a grade of 100% and lose marks for making mistakes. Penalizing mistakes is not my educational style. Indeed, I frame things in exactly the opposite way: students start out with 0% and earn positive marks for every piece of physics logic that they get right. Awarding partial credit fairly is one of my central grading principles, so please **show your working clearly**.

As mentioned in class, I encourage you to discuss *generalities* of the homework problems with other students taking this course for credit, if you like. However, these homeworks are graded individually, so the TA and I need to see specific evidence of individual understanding of each problem (regulations require us to check). All homework assignments must be accompanied by a **completed, signed Academic Integrity Declaration form** or they will not be graded.

I recommend that you handwrite homework assignments, unless LaTeXing them is less laborious for you. Turn in the hardcopy to me directly, or email me the PDF document. Take scans/photos of your assignment before handing in, so that you have your own copy for insurance. If you wish to have the option of requesting a regrade later on, you must submit your work in (indelible) **blue or black pen**. This rule will also be in effect for the midterm and final exams.

1.1: Maxwell's equations and Lorentz transformations of electric/magnetic fields *[10 marks]*

In lecture notes, we indicated how the antisymmetric electromagnetic field strength tensor $F_{\mu\nu}$ contains both the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$ in a "3+1 split",

$$F_{0i} = +\delta_{ij} E^j \,, \qquad F_{ij} = -\mathfrak{E}_{ijk} B^k \,, \tag{1}$$

where Greek indices run from 0 to 3 while Latin ones run from 1 to 3. We also noted that the 4-vector current splits as

$$J^0 = \rho \,, \quad J^i = (\vec{j})^i \,. \tag{2}$$

(a) *[3]* Starting from the above and the covariant Maxwell equation with a source,

$$\partial_\mu F^{\mu\nu} = J^\nu \,, \tag{3}$$

show that you obtain the two 3-vector Maxwell equations involving $\vec{E}$, $\vec{B}$, ρ, and $\vec{j}$. (Hint: study the $\nu = 0$ and $\nu = i$ components of this equation separately.)

(b) *[3]* Show that the covariant Bianchi identity,

$$\mathfrak{E}^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0 \,, \tag{4}$$

produces the other two (sourceless) Maxwell's equations in 3-vector form. (Hint: you will need identites in lecture notes for contracting two $\mathfrak{E}$s. It will also help to notice that $\mathfrak{E}_{0ijk}$ is numerically equal to $\mathfrak{E}_{ijk}$.)

(c) *[4]* By applying the tensorial transformation law for $F_{\mu\nu}$ under a boost in the x^1 direction, find out how $\vec{E}$ and $\vec{B}$ transform under an x^1 boost. Using these rules, show that $|\vec{B}|^2 - |\vec{E}|^2$ is invariant under boosts. Is this quantity also invariant under rotations? Is $\vec{E} \cdot \vec{B}$ invariant under boosts and rotations? Finally, show that you can rewrite your boost transformation rules for $\vec{E}, \vec{B}$ in terms of their components parallel and perpendicular to the boost direction, $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$ and $\vec{B} = \vec{B}_\parallel + \vec{B}_\perp$. You should obtain

$$\begin{aligned}
\vec{E}'_\parallel &= \vec{E}_\parallel \,, \\
\vec{B}'_\parallel &= \vec{B}_\parallel \,, \\
\vec{E}'_\perp &= \gamma_v \left(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp \right) \,, \\
\vec{B}'_\perp &= \gamma_v \left(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp \right) \,,
\end{aligned} \tag{5}$$

where

$$\gamma_v = \frac{1}{\sqrt{1 - v^2}} \,. \tag{6}$$

1.2: Solving the twin paradox for constant relativistic acceleration *[6 marks]*

Pretend that an astronaut twin and a homebody twin are alone in the universe, and that $G_N = 0$, so that the fabric of spacetime is flat. Assume further that you are working in one spatial dimension. Model the astronaut twin's trip as having four parts involving constant relativistic acceleration (CRA):-

1. a CRA rocket burn with $+g$ producing maximum velocity $+v_*$, followed by
2. a CRA rocket burn with $-g$ to reach the turnaround point;
3. a CRA rocket burn with $-g$ producing maximum velocity $-v_*$, followed by
4. a CRA rocket burn with $+g$ to get back home.

By recruiting ideas from lecture notes on CRA, find a formula for how much younger the astronaut twin is when they return home than their homebody twin, as a function of g and v_*, and c (which you can restore using dimensional analysis). Assuming experimentally optimistic values for g and v_*, what order of magnitude of anti-ageing could present-day rocket entrepreneurs offer wealthy celebrities willing to spend one week in a spaceship?

1.3: Flat spacetime in curvilinear coordinates, Christoffels, and Maxima *[9 marks]*

Consider Minkowski spacetime in spherical polar coordinates,

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \,. \tag{7}$$

Even though spacetime is flat, there is a nontrivial affine connection, because the coordinates are curvilinear.

(a) *[3]* Compute the Christoffel symbols in spherical polar coordinates by hand, using either the basis vector method

$$\partial_\lambda \boldsymbol{e}_\mu = \Gamma^\nu{}_{\mu\lambda} \boldsymbol{e}_\nu \,, \qquad g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu \,, \tag{8}$$

or by taking derivatives of the metric tensor

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\nu\sigma} - \partial_\sigma g_{\nu\lambda} \right) \,. \tag{9}$$

(b) *[3]* Find the Christoffels using Maxima. Instructions:-

- If you have access to a computer on which you can install software, download Maxima and install it (preferred). If you do not, you can use Maxima Online. (Note: only Maxima may be used. Work submitted in Maple, Mathematica, Matlab, or other software will not be graded. We are standardizing on Maxima for HW1 and future homeworks in order to make the playing field level for all students.)
- You must hand in your Maxima file, or an equivalent full record of what you did in Maxima Online, with your homework assignment. Be sure to document what the commands you used actually do, as well as showing the output results.
- If you use someone else's Maxima file as a template for your calculations, you must acknowledge that with a proper citation. Academic citation hygiene requires it.

(Hint: here are some sample Maxima files for cylindrical polars: interactive version, command line version.)

(c) *[3]* Using the Christoffel symbols from (a), find the covariant Laplacian of a scalar field $\Psi(t, r, \theta, \phi)$ in spherical polar coordinates, namely $\nabla^\mu \nabla_\mu \Psi$. You should find the resulting expression familiar from multivariable calculus class. (Hint: on a scalar field Ψ, and only on a scalar field, $\nabla_\mu \Psi = \partial_\mu \Psi$.)