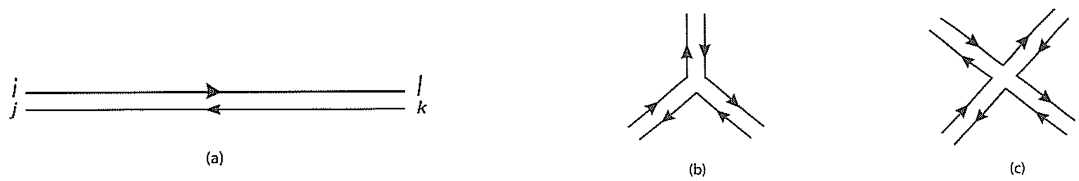
QFT2 (2018) HW3. Due Mar.12. Format: PDF (scan handwriting). Show your working.

Q1: large-N_c

As we know from HW2, the Yang-Mills gauge field lives in the adjoint representation of the gauge group. For the particular case of $U(N_c)$, the adjoint can be represented in terms of a two-index tensor, i.e. a matrix. The propagator behaves like

$$\langle 0|T\{A^A_\mu(x)A^B_\nu(0)\}|0\rangle (T^A)^I{}_J(T^B)^K_L \quad \propto \quad \delta^{AB}(T^A)^I{}_J(T^B)^K_L \quad \propto \quad \delta^I{}_L\delta^K{}_J \,.$$

This motivates the introduction of *double line notation*, in which the gauge information in the Yang-Mills field is represented in Feynman graphs by (oriented) double lines rather than single wavy lines. The propagator and vertices take the form



QCD is famously strongly coupled at low energy, making perturbation theory in g^2 quite useless. Suppose that we take the rank N_c of the gauge group to become parametrically large, without altering matter couplings. The hope is that $1/N_c$ provides a perturbative handle on the physics, even for QCD where $1/N_c$ is only 1/3. Specifically, we take $N_c \to$ large, $\lambda \equiv g^2_{\rm YM}N_c =$ fixed, where λ is known as the 't Hooft coupling.

(a) Feynman diagrams in double line notation can be *planar*, in the sense that you can draw them on the paper without lifting your pen, or they can be *non-planar*. Draw about ten different examples of valid tree and loop diagrams for pure $U(N_c)$ QCD, both planar and non-planar. Figure out how many powers of λ and how many powers of $1/N_c$ are attached to each example Feynman graph, by focusing on group theory. Use the normalization convention in which the Yang-Mills Lagrangian is $\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})/(2g^2_{\rm YM})$.

(b) Focus on vacuum-to-vacuum graphs with no external legs. From any given double-line graph, explain why you can form an orientable 2D surface by sending loops to faces, propagators to edges, and vertices to vertices. By carefully counting the dependence on the number of index loops L, propagators P, and vertices V in any given graph, show that its $1/N_c$ scaling depends only on the Euler characteristic χ of the graph. From the general patterns you see, for vacuum graphs and graphs with external legs, argue that loop corrections in the Feynman graph expansion are organized in a *double perturbation expansion* in powers of λ and powers of $1/N$.

(c) Now add in N_f flavours of minimally coupled quarks. These live in the fundamental (vector) representation rather than in the adjoint, and so they are represented by a single line rather than a double line in the above notation conventions. Work out their pattern for λ, $1/N_c$, and N_f/N_c scaling.

(continued overleaf) $\longrightarrow$

Q2: ghosts in R_ξ gauges

Consider an Abelian gauge field A_μ coupled to a complex scalar field $\Phi = \frac{1}{\sqrt{2}}\left(\phi^1 + i\phi^2\right)$ with

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + |D_\mu\Phi|^2 - V(\Phi)$$

(a) Write down the infinitesimal form of the local $U(1)$ gauge symmetry.

(b) Take the potential $V(\Phi)$ to have the sombrero form we encountered earlier in the course. Suppose that $U(1)$ spontaneously breaks, with Φ developing a vev along (say) the 1 direction. Expand

$$\phi^1(x) = v + h(x)\,, \qquad \phi^2(x) = \varphi(x)\,.$$

Choose the following gauge-fixing function (instead of, say, the Lorentz gauge) for this $U(1)$ symmetry:

$$G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu - \xi\, e\, v\, \varphi\right)$$

Using this specific form of G, derive the Feynman rules (propagators and vertices) for all fields. Show in particular that, although the ghost and antighost decouple from the gauge field, they do *not* decouple from the Higgs.

Note: A generalization of this R_ξ gauge to the non-Abelian gauge theory of the Standard Model of particle physics is used to prove its renormalizability. See Peskin & Schroeder §21 or Weinberg §17 (volume II) for more details.

Q3: 1-loop divergences of scalar Yang-Mills

Write down the Lagrangian for a Yang-Mills gauge field A_μ minimally coupled to a complex scalar field Φ in the fundamental representation. Use it to find the vertices and propagators at tree level. Do not choose any particular gauge group. Then pick any one-loop diagram you can build using your Feynman rules, and write down the expression for it as an integral over loop momentum k. Estimate how this one-loop diagram will blow up in the UV but do not try to evaluate it explicitly. Be sure to make all your gauge indices explicit.
