HW1 Q3: Playing with $SU(3)$

This problem is mostly about learning how to recruit a computer algebra engine to automate boring analytical calculations. Please use only `Maxima` *(free, open source),* `Mathematica`*, or* `Maple`*. Document your code clearly so that I know you authored it.*

Consider the Lie group $SU(3)$ pertinent to QCD. In the fundamental representation f with 3-component column vectors representing red, green and blue quarks, the generators in the standard Gell-Mann basis are

$$\begin{aligned} T^1 &= \frac{1}{2}\begin{bmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T^4 = \frac{1}{2}\begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix} \quad T^7 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix} \\ T^2 &= \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T^5 = \frac{1}{2}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{bmatrix} \quad T^8 = \frac{1}{2\sqrt{3}}\begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\ T^3 &= \frac{1}{2}\begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T^6 = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{bmatrix} . \end{aligned} \tag{1}$$

(a) Explain why the generators of $su(3)$ in eq.(1) are traceless and Hermitean and why there are eight of them. Explain which ones form the Cartan subalgebra of $su(3)$.

(b) In general, for any representation r,

$$\mathrm{Tr}\left\{T^A T^B\right\} = \mathscr{C}(r)\delta^{AB} . \tag{2}$$

Show by explicit computation from the generators in eq.(1) that $\mathscr{C}(f) = \frac{1}{2}$.
Do a couple of representative examples by hand to check that your code works properly, then automate the rest of the boring algebra with code.

(c) Starting from the normalization condition $\mathrm{Tr}\left(T_r^A T_r^B\right) = \mathscr{C}(r)\delta^{AB}$ and the Lie algebra commutation relations $[T^A, T^B] = i f^{ABC} T^C$ for any representation, show that

$$f^{ABC} = -\frac{i}{\mathscr{C}(r)}\mathrm{Tr}\left([T_r^A, T_r^B]T_r^C\right) . \tag{3}$$

Use this formula to compute the structure constants f^{ABC} for $su(3)$ using the explicit form of the generators in eq.(1).
Automate the boring algebra with code.

(d)* For bonus points, show that for the adjoint representation $\mathscr{C}(Adj) = N$.
The ingredients needed are eq.(2), $(T^A_{Adj})^{BC} = -i f^{ABC}$, and code.

Reminder: your entire assignment including code worksheets must fit within 25 pages.