PHY2404S (2018) HW instructions:-
Show your working: I am not telepathic.
Total page count for any homework assignment may not exceed 25.
Required format: PDF, submitted via email. Scan in handwritten answers.

HW1 Q1 of 3: Algebra of Poincaré Group

(a) Working in arbitrary dimension D, consider the generators of total angular momentum

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu} = -X_\mu P_\nu + X_\nu P_\mu + \Sigma_{\mu\nu} \tag{1}$$

where $L_{\mu\nu}$ encodes orbital angular momentum and $\Sigma_{\mu\nu}$ encodes spin angular momentum. Using this definition and the canonical commutation relations for position and momentum, derive the commutation relations for the Poincaré algebra:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \\ [P_\mu, M_{\rho\sigma}] &= +i\left(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\right) \\ [M_{\mu\nu}, M_{\rho\sigma}] &= +i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\right) \end{aligned} \tag{2}$$

Be explicit about each assumption you need to use along the way. In particular, state what you assume about the commutators of spin generators $\Sigma_{\mu\nu}$ with X_μ and P_ν and why. Your argument should work for any D and any mass m and spin s.

(b) Specialize to $D = 1+3$. Defining $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and $M_{0i} = K_i$, show that

$$\begin{aligned} [K_i, K_j] &= -i\epsilon_{ijk}J_k \\ [J_i, K_j] &= +i\epsilon_{ijk}K_k \\ [J_i, J_j] &= +i\epsilon_{ijk}J_k\,. \end{aligned} \tag{3}$$

Further, show that defining $N_i = a(J_i + iK_i)$ and $N_i^\dagger = a(J_i - iK_i)$ for real a gives

$$\begin{aligned} [N_i, N_j] &= +i\epsilon_{ijk}N_k \\ \left[N_i^\dagger, N_j^\dagger\right] &= +i\epsilon_{ijk}N_k^\dagger \\ \left[N_i, N_j^\dagger\right] &= 0 \end{aligned} \tag{4}$$

as long as $a = \frac{1}{2}$.